SIGNAL RECOVERY, BOURGAIN'S Λ_q PROBLEM, AND MULTILINEAR RESTRICTIONS

TRAN DUY ANH LE

Abstract. In this paper, we work on multilinear restriction and its applications on signal recovery. Specifically, we will show that sending more signals will have a higher chance of recovering the original information. We also prove a generalization of Bourgain's result on Λ_q problem for multiple functions and apply it to multiple transmissions.

Contents

1. Introduction 1
2. Notation and Preliminaries 2
3. Multilinear restriction and its applications. 7
4. Bourgain's Λ_q problem 18
Acknowledgement 21
References 21

1. Introduction

Information transportation has been an important aspect of real life, where we send the signals between people. In some cases, there will be noises so there will be a chance that we lost some of the information that was sent out. Donoho and Stark [2] have proved that although some of the frequencies may be lost, we can still recover the signal uniquely by Discrete Fourier Transform.

There are also questions regarding how restriction theory affects signal recovery by giving estimates on the restriction of Uncertainty principles. Moreover, we want to work on sending multiple copies of a signal to recover uniquely the original one. One can ask whether we can improve the recovering algorithms are loosening the Uncertainty Principles, i.e the size of the lost data is larger. We also discuss some drawbacks regarding the computational aspects of the algorithm.

In this paper, Section 2 will provide basic preliminaries and notations that will be used throughout the paper. Section 3 will be the core of the paper, where we discuss multilinear restrictions and its applications to signal recovery. We conclude with section 4 for a generalization of Bourgain's Λ_q problem and its applications.

2. Notation and Preliminaries

Firstly, we define what a signal is.

Definition 2.1 (Signal)**.** A signal f is a function that takes value in $\mathbb{Z}_N^d$ and maps to $\mathbb{C}$.

For any $f : \mathbb{Z}_N^d \to \mathbb{C}$, we define the Fourier transform to be

$$\widehat{f}(m) = N^{\frac{-d}{2}} \sum_x \chi(-x \cdot m) f(x)$$

where $\chi(t) = e^{\frac{2\pi it}{N}}$.

From this definition, one can see that the Inverse Fourier Transform is given by

$$\overset{\vee}{f}(x) = \sum_m \chi(x \cdot m) f(m).$$

To see why this is true, we will apply the Inverse Fourier Transform to the Fourier Transform. We have

$$\begin{aligned} N^{\frac{-d}{2}} \sum_m \chi(x \cdot m) \widehat{f}(m) &= N^{\frac{-d}{2}} \sum_m \chi(x \cdot m) N^{\frac{-d}{2}} \sum_{x'} \chi(-x' \cdot m) f(x') \\ &= N^{-d} \sum_m \sum_{x'} \chi((x - x') \cdot m) f(x') = f(x). \end{aligned}$$

We also define what a support of a signal is.

Definition 2.2 ($\mathrm{spt}(f)$)**.** The support of a signal f, denoted as $\mathrm{spt}(f)$, is

$$\mathrm{spt}(f) = \{x \in \mathbb{Z}_N^d : f(x) \neq 0\}.$$

One of the key property of Fourier transform is that it preserves the L_2-norm of the function. This is called the Plancherel's formula.

Theorem 2.3 (Plancherel's theorem)**.** *For any signal* $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\sum_{m \in \mathbb{Z}_N^d} |\widehat{f}(m)|^2 = \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

Proof. We have

$$\begin{aligned}
\sum_{m\in\mathbb{Z}_N^d} |\widehat{f}(m)|^2 &= \sum_{m\in\mathbb{Z}_N^d} \left|N^{-\frac{d}{2}}\sum_x \chi(-x\cdot m)f(x)\right|^2 \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d}(\sum_x \chi(-x\cdot m)f(x))\overline{(\sum_x \chi(-x\cdot m)f(x))} \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d}\left(\sum_x \chi(-x\cdot m)f(x)\right)\left(\sum_x \chi(x\cdot m)\overline{f(x)}\right) \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d}\sum_{x_1,x_2} \chi((x_2-x_1)\cdot m)f(x_1)\overline{f(x_2)} \\
&= N^{-d}\sum_{x_1,x_2} f(x_1)\overline{f(x_2)}\sum_{m\in\mathbb{Z}_N^d}\chi((x_2-x_1)\cdot m).
\end{aligned}$$

By Gauss sum, we have for any $x\in\mathbb{Z}_N^d$,

$$\sum_{m\in\mathbb{Z}_N^d}\chi(x\cdot m) = \begin{cases} N^d & \text{If } x=\mathbf{0} \\ 0 & \text{Otherwise} \end{cases}.$$

Hence

$$\sum_{m\in\mathbb{Z}_N^d}|\widehat{f}(m)|^2 = N^{-d}N^d\sum_x f(x)\overline{f(x)} = \sum_x |f(x)|^2.$$

Given a set S and suppose $\{\widehat{f}(m)\}_{m\in S}$ are lost. Note that we can construct multiple signals so that its Fourier Transform aligns with the data that isn't lost, i.e the points $m\notin S$. However, one can ask whether the recovered signal is unique or not. We have the following principle.

Theorem 2.4 (Uncertainty Principle [4]). *If f is supported in E, $\widehat{f}$ is supported in S, then $|E|\cdot|S|\geq N^d$.*

Proof. From the inverse formula

$$f(x) = N^{-\frac{d}{2}}\sum_{m\in S}\chi(x\cdot m)\widehat{f}(m) = N^{-d}\sum_{m\in S}\chi(x\cdot m)\sum_{x'\in E}\chi(x'\cdot m)f(x').$$

Then

$$\begin{aligned}
|f(x)| = N^{-d}\left|\sum_{m\in S}\chi(x\cdot m)\sum_{x'\in E}\chi(x'\cdot m)f(x')\right| &\leq N^{-d}\sum_{m\in S}\left|\sum_{x'\in E}\chi(x'\cdot m)f(x')\right| \\
\leq N^{-d}\sum_{m\in S}\sum_{x'\in E}|f(x')| = N^{-d}|S|\sum_{x'\in E}f(x').
\end{aligned}$$

Hence
$$\sum_{x\in E}|f(x)| \le \sum_{x\in E} N^{-d}|S|\sum_{x'\in E} f(x') = N^{-d}|E|\cdot|S|\sum_{x'\in E}|f(x)| \implies |E|\cdot|S|\ge N^d.$$

We have the definition for a set to be (p,q)-restricted.

Definition 2.5 ((p,q)-Restriction [4]). Let $S\subseteq \mathbb{Z}_N^d$. We say S satisfies a (p,q)-restriction if there exists a uniform constant $C_{p,q}$ such that for any $f:\mathbb{Z}_N^d\to\mathbb{C}$
$$\left(\frac{1}{|S|}\sum_{m\in S}|\widehat{f}(m)|^q\right)^{\frac{1}{q}} \le C_{p,q}N^{-\frac{d}{2}}\left(\sum_{x\in\mathbb{Z}_N^d}|f(x)|^p\right)^{\frac{1}{p}}.$$

For simplicity in calculations, we will define what an L_p norm on a set. In this case, we want to take account with the counting measure.

Definition 2.6 ($\|\cdot\|_{L^p(A)}$). Given $f:\mathbb{Z}_N^d\to\mathbb{C}$ and a set $A\subseteq\mathbb{Z}_N^d$, define
$$\|f\|_{L^p(A)} = \left(\sum_{x\in A}|f(x)|^p\right)^{\frac{1}{p}}, \|f\|_{L^p(\mu_A)} = \left(\frac{1}{A}\sum_{x\in A}|f(x)|^p\right)^{\frac{1}{p}}.$$

Similarly
$$\|f\|_{L^p(\mu)} = \left(\frac{1}{N^d}\sum_{x}|f(x)|^p\right)^{\frac{1}{p}}.$$

Now, we start with the first condition for a signal to be uniquely recovered.

Theorem 2.7. *Let $f:\mathbb{Z}_N^d\to\mathbb{C}$ and suppose $\{\widehat{f}(m)\}_{m\in S}$ are lost. If $|S|\cdot|spt(f)|<\frac{N^d}{2}$, then we can reconstruct f uniquely.*

Proof. Suppose f cannot be recovered uniquely, then there exists $g:\mathbb{Z}_N^d\to\mathbb{C}$ such that $|\mathrm{spt}(f)|=|\mathrm{spt}(g)|$ and $\widehat{f}=\widehat{g}$ away from S. Let $h=f-g$, then
$$|\mathrm{spt}(h)|\le 2|\mathrm{spt}(f)|, |\mathrm{spt}(\widehat{h})|\le|S|.$$

By Uncertainty Principle, we get
$$2|S|\cdot|\mathrm{spt}(f)|\ge|\mathrm{spt}(h)|\cdot|\mathrm{spt}(\widehat{h})|\ge N^d \implies |S|\cdot|\mathrm{spt}(f)|\ge\frac{N^d}{2},$$
a contradiction. Hence the reconstruction is unique.

Now, we will develop analogous variations to the Uncertainty principle if we assume (p,q)-restriction condition.

Theorem 2.8. *Suppose $f:\mathbb{Z}_N^d\to\mathbb{C}$ is supported in $E\subseteq\mathbb{Z}_N^d$, and $\widehat{f}:\mathbb{Z}_N^d\to\mathbb{C}$ is supported in S. Suppose that the restriction estimation holds for a pair (p,q), $1\le p\le q<\infty$. Then*
$$|E|^{\frac{1}{p}}|S|\ge\frac{N^d}{C_{p,q}}.$$

Proof. Suppose f is supported in E and $\widehat{f}$ is supported in S. Then

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m).$$

By Holder's inequality,

$$\begin{aligned}
|f(x)| &= N^{-\frac{d}{2}} \left| \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m) \right| \\
&\leq N^{-\frac{d}{2}} \left(\sum_{m \in S} |\chi(x \cdot m)|^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \\
&= N^{-\frac{d}{2}} |S|^{\frac{q-1}{q}} \left(\sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \\
&= N^{-\frac{d}{2}} |S| \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \leq C_{p,q} N^{-d} |S| \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$|f(x)|^p \leq C_{p,q}^p N^{-pd} |S|^p \sum_{x \in E} |f(x)|^p$$

so by summing all over, we have

$$1 \leq C_{p,q}^p N^{-pd} |E| \cdot |S|^p \implies |S| \cdot |E|^{\frac{1}{p}} \geq \frac{N^d}{C_{p,q}}.$$

Theorem 2.9. *Suppose $f : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in $E \subseteq \mathbb{Z}_N^d$, and $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in S. Suppose that the restriction estimation holds for a pair (p, q), $1 \leq p \leq 2 \leq q < \infty$. Then*

$$|E|^{\frac{2-p}{p}} |S| \geq \frac{N^d}{C_{p,q}^2}.$$

Proof. First of all, we see that as $2 \leq q$, $\|f\|_{L^2(\mu_A)} \leq \|f\|_{L^q(\mu_A)}$ by the generalized Holder's inequality. Hence

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \leq C_{p,q} N^{-\frac{d}{2}} \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

By Holder's inequality, we have

$$C_{p,q} N^{-\frac{d}{2}} \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}} \leq C_{p,q} |E|^{\frac{1}{p} - \frac{1}{2}} N^{-\frac{d}{2}} \left(\sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}$$

By Plancherel's equality, we have

$$\left(\frac{1}{|S|}\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}} = |S|^{-\frac{1}{2}}\left(\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}} = |S|^{-\frac{1}{2}}\left(\sum_{x\in E}|f(x)|^2\right)^{\frac{1}{2}}.$$

so

$$|S|^{-\frac{1}{2}}\left(\sum_{x\in E}|f(x)|^2\right)^{\frac{1}{2}} \le C_{p,q}N^{-\frac{d}{2}}|E|^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{x\in E}|f(x)|^2\right)^{\frac{1}{2}} \iff |S|^{-\frac{1}{2}} \le C_{p,q}N^{-\frac{d}{2}}|E|^{\frac{1}{p}-\frac{1}{2}}$$
$$\iff |E|^{\frac{2-p}{p}}|S| \ge \frac{N^d}{C_{p,q}^2}$$

Corollary 2.10. *If the constant $C_{p,q}$ is small, then the second bound is strictly stronger than the first one.*

Now, we will consider how we can actually recover the signal. We consider Logan's algorithm or L_1-minimization method [3].

Definition 2.11 (Logan's algorithm)**.** We let $g = \arg\min_u \|u\|_1$ with $\widehat{u}(m) = \widehat{f}(m)$ for any $m \notin S$.

To see how this algorithm works, let $f = g + h$ where $h : \mathbb{Z}_N^d \to \mathbb{C}$ and f is supported in E. Then

$$\|g\|_1 = \|f-h\|_1 = \|f-h\|_{L^1(E)} + \|h\|_{L^1(E^c)} \ge \|f\|_{L^1(E)} - \|h\|_{L^1(E)} + \|h\|_{L^1(E^c)}.$$

We have for any x

$$h(x) = \sum_{m\in S} N^{-d/2}\chi(x\cdot m)\widehat{h}(m) = N^{-d}\sum_{m\in S}\sum_{\theta}\chi(x\cdot m)\chi(-\theta\cdot m)h(\theta) \implies |h(x)| \le N^{-d}|S|\|h\|_{L^1(\mathbb{Z}_N^d)}.$$

Summing over $x \in E$, we get

$$\|h\|_{L^1(E)} \le \frac{|E||S|}{N^d}\|h\|_1 = \frac{|E||S|}{N^d}(\|h\|_{L^1(E)} + \|h\|_{L^1(E^c)})$$

and we get $\left(1 - \frac{|E||S|}{N^d}\right)\|h\|_{L^1(E)} \le \frac{|E||S|}{N^d}\|h\|_{L^1(E^c)}$. Note that if we assume that $|S||E| < \frac{N^d}{2}$ so that the result signal is unique, then $\frac{|E||S|}{N^d} < \frac{1}{2}$ or we get $\|h\|_{L^1(E^c)} > \|h\|_{L^1(E)}$. Hence $\|g\|_1 > \|f\|_1$, contradicting the fact $\|g\|_1$ is minimized.

Therefore, we get $h \equiv 0$ or the algorithm returns the exact signal.

Moreover, we also consider the L_2-minimization method, or Donoho-Stark's algorithm [3].

Definition 2.12 (Donoho-Stark's algorithn)**.** We let $g = \arg\min_u \|u\|_2$ with $\widehat{g}(m) = \widehat{f}(m)$ for any $m \notin S$ and $|\mathrm{spt}(u)| = |\mathrm{spt}(f)|$.

Suppose that $f = g + h$ for some $h : \mathbb{Z}_N^d \to \mathbb{C}$. Then h is supported in T such that $|T| \le 2|E|$. Then $\|h\|_2 = \|h\|_{L^2(T)}$. Moreoer

$$h(x) = N^{-\frac{d}{2}}\sum_{m\in S}\chi(x\cdot m)\widehat{h}(m) \implies |h(x)| \le N^{-\frac{d}{2}}\sum_{m\in S}|\widehat{h}(m)| \le N^{-\frac{d}{2}}|S|^{\frac{1}{2}}\|h\|_{L^2(T)}.$$

Hence, summing over all elements in T, we have

$$\|h\|^2_{L^2(T)} \leq |T|N^{-d}|S|\|h\|^2_{L^2(T)} \implies \sqrt{\frac{|S||T|}{N^d}} \geq 1.$$

Since $|T| \leq 2|E|$, we get $|E||S| \geq \frac{N^d}{2}$, a contradiction to the Uncertainty principle.

Therefore, we get $h \equiv 0$ or the algorithm returns the exact signal. However, this algorithm takes exponential time.

3. Multilinear restriction and its applications.

One natural question to be asked is that by sending more information, do we have a higher chance of recovering the original signal? In this section, we will show that it can loosen the recovery condition.

First of all, we introduce a bilinear restriction result.

Theorem 3.1 (Bilinear restriction)**.** *Let $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ such that $\widehat{f}$ is supported in X and $\widehat{g}$ is supported in Y. Then*

$$\|fg\|_{L^2(\mu)} \leq \|1_X * 1_Y\|_\infty^{\frac{1}{2}}\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}.$$

Proof. We have the following lemma.

Lemma 3.2 (Riesz-Thorin interpolation theorem [5])**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Let $p_0, p_1 \in [1, \infty]$ and for $0 < t < 1$, define p_t by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Then if $T : (L^{p_0} + L^{p_1}) \to (L^{p_0} + L^{p_1})$ is a linear map such that for $f_0 \in L^{p_0}(X, \mathcal{A}, \mu)$ and $f_1 \in L^{p_1}(X, \mathcal{A}, \mu)$, then

$$\|Tf_0\|_{p_0} \leq M_0\|f_0\|_{p_0}, \|Tf_1\|_{p_1} \leq M_1\|f_1\|_{p_1}$$

where $M_0, M_1 \in (0, \infty)$. Then

$$\|Tf\|_{p_t} \leq M_0^{1-t}M_1^t\|f\|_{p_t}.$$

We also have if $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ and h be given by $h(x) = (f * g)(x) = \sum_y f(y)g(x-y)$, then

$$\begin{aligned}
\widehat{h}(m) &= N^{-\frac{d}{2}}\sum_x \chi(-x\cdot m)h(x) = N^{-\frac{d}{2}}\sum_x \chi(-x\cdot m)\sum_y f(y)g(x-y) \\
&= N^{-\frac{d}{2}}\sum_a\sum_b \chi(-(a+b)\cdot m)f(a)g(b) = N^{-\frac{d}{2}}\sum_a\sum_b \chi(-a\cdot m)f(a)\chi(-b\cdot m)g(b) \\
&= N^{\frac{d}{2}}\widehat{f}(m)\widehat{g}(m)
\end{aligned}$$

so we get $\widehat{fg} = N^{-\frac{d}{2}}(\widehat{f} * \widehat{g})$. Back to the problem, we have

$$\begin{aligned}\|fg\|_{L^2(\mu)} &= \left(\frac{1}{N^d}\sum_x |f(x)g(x)|^2\right)^{\frac{1}{2}} = \left(\frac{1}{N^d}\sum_m |\widehat{fg}(m)|^2\right)^{\frac{1}{2}} = \left(\frac{1}{N^d}\sum_m |N^{-\frac{d}{2}}(\widehat{f} * \widehat{g})|^2\right)^{\frac{1}{2}}\\ &= \left(N^{-2d}\sum_m |\widehat{f} * \widehat{g}|^2\right)^{\frac{1}{2}} = N^{-d}\left(\sum_m |\widehat{f}1_X * \widehat{g}1_Y|^2\right)^{\frac{1}{2}} = N^{-d}\|\widehat{f}1_X * \widehat{g}1_Y\|_2.\end{aligned}$$

Now, we have

$$\begin{aligned}\|\widehat{f}1_X * \widehat{g}1_Y\|_1 &= \sum_{m_1} |\widehat{f}1_X * \widehat{g}1_Y| = \sum_{m_1}\left|\sum_{m_2} \widehat{f}1_X(m_1 - m_2)\widehat{g}1_Y(m_2)\right|\\ &\le \sum_{m_1}\sum_{m_2} |\widehat{f}1_X(m_1 - m_2)||\widehat{g}1_Y(m_2)|\\ &\le \sum_{m_1} |\widehat{f}1_X(m_1)|\sum_{m_2} |\widehat{g}1_Y(m_2)| \le \|\widehat{f}\|_1\|\widehat{g}\|_1.\end{aligned}$$

On the other hand

$$\|\widehat{f}1_X * \widehat{g}1_Y\|_\infty \le \|\widehat{f}\|_\infty\|\widehat{g}\|_\infty\|1_X * 1_Y\|_\infty.$$

Hence, by the Riesz-Thorin interpolation theorem,

$$\begin{aligned}\|fg\|_{L^2(\mu)} &= N^{-d}\|\widehat{f}1_X * \widehat{g}1_Y\|_2 \le N^{-d}\|1_X * 1_Y\|_\infty^{\frac{1}{2}}\|\widehat{f}\|_2\|\widehat{g}\|_2 = N^{-d}\|1_X * 1_Y\|_\infty^{\frac{1}{2}}\|f\|_2\|g\|_2\\ &= \|1_X * 1_Y\|_\infty^{\frac{1}{2}}\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}.\end{aligned}$$

Therefore, the proof is completed.

We also want to generalize the Riesz-Thorin interpolation to n functions.

Definition 3.3 (n-linear map)**.** A function $f : V_1 \times V_2 \times \ldots \times V_n \to W$ is a n-linear map such that if we fixed any entry of the function, then we get a $n-1$ linear map.

For example, a bilinear map $f : V_1 \times V_2 \to W$ is a map such that for any v_1, the map $f_{v_1} : v_2 \mapsto f(v_1, v_2)$ is a linear map and for any v_2, the map $f_{v_2} : v_1 \mapsto f(v_1, v_2)$ is a linear map.

We have the following generalization.

Lemma 3.4 (Generalization of Riesz-Thorin interpolation theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Let $p_0, p_1 \in [1, \infty]$ and for $0 < t < 1$, define p_t by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Then if $T : (L^{p_0} + L^{p_1})^n \to (L^{p_0} + L^{p_1})$ is a n-linear map such that for $f_1, f_2, \ldots, f_n \in L^{p_0}(X, \mathcal{A}, \mu)$ and $g_1, g_2, \ldots, g_n \in L^{p_1}(X, \mathcal{A}, \mu)$, then

$$\|T(f_1, \ldots, f_n)\|_{p_0} \le M_0 \prod_{i=1}^n \|f_i\|_{p_0}, \|T(g_1, \ldots, g_n)\|_{p_1} \le M_1 \prod_{i=1}^n \|g_i\|_{p_1}$$

where $M_0, M_1 \in (0, \infty)$. *Then*

$$\|T(f_1, \dots, f_n)\|_{p_t} \le M_0^{1-t} M_1^t \prod_{i=1}^n \|f_i\|_{p_t}.$$

Proof. We will work on the case of bilinear. Let $B : (L^{p_0} + L^{p_1})^n \to (L^{p_0} + L^{P_1})$ be a bilinear map that satisfy the upper condition. Fix $f_1^*, f_2^* \in L^{p_0} + L^{p_1}$ and define

$$T_{f_1^*} f_2(x) = B(f_1^*, f_2)(x), T_{f_2*} f_1(x) = B(f_1, f_2^*)(x).$$

Then we have for any $i \in \{0, 1\}$

$$\|T_{f_1^*}\|_{p_i} = \inf\{c \ge 0 : \|T_{f_1^*} f_2\|_{p_i} \le c\|f_2\|_{p_i} \forall f_2 \in L^{p_0} + L^{p_1}\} = \inf\{c \ge 0 : \|B(f_1^*, f_2)\|_{p_i} \le c\|f_2\|_{p_i}\}.$$

Similarly

$$\|T_{f_2^*}\|_{p_i} = \inf\{c \ge 0 : \|B(f_1, f_2^*)\|_{p_i} \le c\|f_1\|_{p_i}\}.$$

We have

$$\|B(f_1, f_2^*)\|_{p_i} \le M_i \|f_1\|_{p_0} \|f_2^*\|_{p_0} \implies \frac{\|B(f_1, f_2^*)\|_{p_i}}{\|f_1\|_{p_i}} \le M_0 \|f_2^*\|_{p_i}.$$

Consider $\|f_2^*\|_{p_i}$ as a constant, we get $\|T_{f_2^*}\|_{p_i} \le M_0 \|f_2^*\|_{p_i}$. By Riesz-Thorin interpolation theorem, we get

$$\|T_{f_2^*}\|_{p_t} \le M_0^{1-t} M_1^t \|f_2^*\|_{p_t} \implies \inf\{c \ge 0 : \|B(f_1, f_2^*)\|_{p_t} \le c\|f_1\|_{p_t}\} \le M_0^{1-t} M_1^t \|f_2^*\|_{p_t}.$$

Therefore, for any $f_1 \in L^{p_0} + L^{p_1}$, $\|B(f_1, f_2^*)\|_{p_t} \le M_0^{1-t} M_1^t \|f_2^*\|_{p_t} \|f_1\|_{p_t}$.

Similarly for $T_{f_1^*}$, we get

$$\|T_{f_1^*}\|_{p_t} \le M_0^{1-t} M_1^t \|f_1^*\|_{p_t} \implies \|B(f_1^*, f_2)\|_{p_t} \le M_0^{1-t} M_1^t \|f_1^*\|_{p_t} \|f_2\|_{p_t}.$$

Hence, we get the result.

For the case of n-linear, we do a similar method. Let $T : (L^{p_0} + L^{p_1})^n \to (L^{p^0} + L^{p_1})$ be a n-linear map that satisfy the condition and we assume that the theorem is true for $n-1$-linear functions. Fix $f_1^*, f_2^*, \dots, f_n^* \in L^{p_0} + L^{p_1}$ and for each $1 \le i \le n$,

$$T_i(f_i)(x) = T(f_1^*, f_2^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)(x).$$

Then for such $j \in \{0, 1\}$,

$$\|T_i\|_{p_j} = \inf\{c \ge 0 : \|T(f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)\|_{p_j} \le c\|f_i\|_{p_j} \forall f_i \in L^{p_0} + L^{p_1}\}.$$

We have

$$\|T(f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)\|_{p_j} \le M_j \|f_1^*\|_{p_j} \dots \|f_{i-1}^*\|_{p_j} \|f_i\|_{p_j} \|f_{i+1}\|_{p_j} \dots \|f_n^*\|_{p_j}$$

so we get $\frac{\|T(f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)\|_{p_j}}{\|f_i\|_{p_j}} \le M_j \|f_1^*\|_{p_j} \dots \|f_{i-1}^*\|_{p_j} \|f_{i+1}^*\|_{p_j} \dots \|f_n^*\|_{p_j}$ and $\|T_i\|_{p_j} \le M_j \|f_1^*\|_{p_j} \dots \|f_{i-1}^*\|_{p_j} \|f_{i+1}^*\|_{p_j} \dots \|f_n^*\|_{p_j}$. By inductive hypothesis, we get

$$\|T_i\|_{p_t} \le M_0^{1-t} M_1^t \|f_1^*\|_{p_t} \dots \|f_{i-1}^*\|_{p_t} \|f_{i+1}^*\|_{p_t}.$$

Therefore, for any $f_i \in L^{p_0} + L^{p_1}$,

$$\|T(f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)\|_{p_t} \le M_0^{1-t} M_1^t \|f_1^*\|_{p_t} \dots \|f_{i-1}^*\|_{p_t} \|f_i\|_{p_t} \|f_{i+1}^*\|_{p_t} \dots \|f_n^*\|_{p_t}.$$

Hence, we are done.

Now, we will prove a general result of Theorem 3.1

Theorem 3.5 (Multilinear Restriction). *Let $f_1, f_2, \ldots, f_k : \mathbb{Z}_N^d \to \mathbb{C}$ such that $\widehat{f_i}$ is supported in X_i for any $1 \leq i \leq k$. Then*

$$\|f_1 \ldots f_k\|_{L^2(\mu)} \leq \|1_{X_1} * \ldots * 1_{X_n}\|_\infty^{\frac{1}{2}} \prod_{i=1}^{k} \|f_i\|_{L^2(\mu)}.$$

Proof. Before proving the main theorem, we will generalize the formula for the Fourier transform of a product of functions.

Lemma 3.6. *Given k signals $f_1, f_2, \ldots, f_k : \mathbb{Z}_N^d \to \mathbb{C}$, we have*

$$\widehat{f_1 f_2 \ldots f_k} = N^{\frac{(k-1)d}{2}} (\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_k}).$$

Proof. We will prove the formula by induction. For $k = 1, 2$, we have covered in the upper theorem. Suppose that $\widehat{f_1 f_2 \ldots f_t} = N^{\frac{(t-1)d}{2}} (\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_t})$ for some $t \geq 2$. Then we have

$$\begin{aligned}
\widehat{f_1 f_2 \ldots f_{t+1}} &= N^{\frac{d}{2}} (\widehat{f_1 f_2 \ldots f_t} * \widehat{f_{t+1}}) = N^{\frac{d}{2}} (N^{\frac{(t-1)d}{2}} (\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_t}) * \widehat{f_{t+1}}) \\
&= N^{\frac{td}{2}} (\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_{t+1}}).
\end{aligned}$$

Therefore, the claim is true for $k = t + 1$. Hence, by the Principle of Mathematical Induction, we have the result.

We have

$$\begin{aligned}
\|f_1 f_2 \ldots f_k\|_{L^2(\mu)} &= \left(\frac{1}{N^d} \sum_x |f_1(x) f_2(x) \ldots f_k(x)|^2 \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{N^d} \sum_m |\widehat{f_1 f_2 \ldots f_k}(m)|^2 \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{N^d} \sum_m |N^{-\frac{(k-1)d}{2}} (\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_k})|^2 \right)^{\frac{1}{2}} \\
&= \left(N^{-kd} \sum_m |\widehat{f_1} * \widehat{f_2} * \ldots * \widehat{f_k}|^2 \right)^{\frac{1}{2}} \\
&= N^{-\frac{k}{2}d} \left(\sum_m |\widehat{f_1} 1_{X_1} * \widehat{f_2} 1_{X_2} * \ldots * \widehat{f_k} 1_Y|^2 \right)^{\frac{1}{2}} \\
&= N^{-\frac{k}{2}d} \|\widehat{f_1} 1_{X_1} * \widehat{f_2} 1_{X_2} * \ldots * \widehat{f_k} 1_{X_k}\|_2.
\end{aligned}$$

Now, we have

$$\begin{aligned}\|\widehat{f}_1 1_{X_1} * \widehat{f}_2 1_{X_2} * \ldots * \widehat{f}_k 1_{X_k}\|_1 &\leq \|\widehat{f}_1 1_{X_1}\|_1 \|\widehat{f}_2 1_{X_2} * \ldots * \widehat{f}_k 1_{X_k}\|_1 \\ &\leq \|\widehat{f}_1 1_{X_1}\|_1 \|\widehat{f}_2 1_{X_2}\|_1 \|\widehat{f}_3 1_{X_3} * \ldots * \widehat{f}_k 1_{X_k}\|_1 \\ &\leq \ldots \\ &\leq \prod_{i=1}^{k} \|\widehat{f}_i 1_{X_i}\|_1.\end{aligned}$$

On the other hand

$$\|\widehat{f}_1 1_{X_1} * \ldots * \widehat{f}_k 1_{X_k}\|_\infty \leq \|\widehat{f}_1\|_\infty \|\widehat{f}_2\|_\infty \ldots \|\widehat{f}_k\|_\infty \|1_{X_1} * 1_{X_2} * \ldots * 1_{X_k}\|_\infty.$$

Hence, by the generalization of Riesz-Thorin interpolation theorem, we have

$$\begin{aligned}\|f_1 f_2 \ldots f_k\|_{L^2(\mu)} &= N^{-\frac{k}{2}d} \|\widehat{f}_1 1_{X_1} * \widehat{f}_2 1_{X_2} * \ldots * \widehat{f}_k 1_{X_k}\|_2 \\ &\leq N^{-\frac{k}{2}d} \|1_{X_1} * 1_{X_2} * \ldots * 1_{X_k}\|_\infty^{\frac{1}{2}} \|\widehat{f}_1\|_2 \|\widehat{f}_2\|_2 \ldots \|\widehat{f}_k\|_2 \\ &= \|1_{X_1} * 1_{X_2} * \ldots * 1_{X_k}\|_\infty^{\frac{1}{2}} (N^{-d/2}\|f_1\|_2)(N^{-d/2}\|f_2\|_2) \ldots (N^{-d/2}\|f_k\|_2) \\ &= \|1_{X_1} * 1_{X_2} * \ldots * 1_{X_k}\|_\infty^{\frac{1}{2}} \|f_1\|_{L^2(\mu)} \|f_2\|_{L^2(\mu)} \ldots \|f_k\|_{L^2(\mu)}.\end{aligned}$$

Now, we want to see the applications of multilinear restrictions to signal recovery. We first introduce the idea of a channel.

Definition 3.7 (Channel)**.** Let $n \in \mathbb{N}$. An n-channel is a collection of n repeated signals, denoted as

$$\mathrm{Chan}_n(f) = \{\widehat{f}, \widehat{f}, \ldots, \widehat{f}\}.$$

Then, instead of sending a single signal, we send an n-channel of its Fourier transform. Suppose for each $1 \leq i \leq n$, the values of $\widehat{f}_i(x)$ are lost on the set X_i. We will see if there are any conditions such that we can recover uniquely the signal f.

Theorem 3.8. *Suppose all the values of f_i agrees on a single output, i.e for any $t \in \mathbb{Z}_N^d$*

$$|\{f_i(t) : 1 \leq i \leq n\}| \leq 1.$$

To recover the original signal, we consider the signal $\widehat{F}$, which is not defined on $X_1 \cap \ldots \cap X_n$, with the values taken from $f_1, \ldots, f_n$. If $|X_1 \cap \ldots \cap X_n| \cdot |spt(f)| < \frac{N^d}{2}$, then we can construct f uniquely.

Proof. This follows from Theorem 2.1.

For any $1 \leq i \leq n$, suppose we randomly take X_i uniformly random of size s_i from $\mathbb{Z}_N^d$, then we want to calculate the Expectation of the size of $X_1 \cap \ldots \cap X_n$. For each $t \in \mathbb{Z}_N^d$ and $1 \leq i \leq n$, denote $p_{X_i}(t)$ to be the probability that $t \in X_i$.

Lemma 3.9. *For any $t \in \mathbb{Z}_n^d$ and $1 \leq i \leq n$, $p_{X_i}(t) = \frac{s_i}{N^d}$.*

Proof. We have

$$p_{X_1}(t) = \frac{|\{X \subseteq \mathbb{Z}_N^d : |X| = s_i, t \in X\}|}{|\{X \subseteq \mathbb{Z}_N^d : |X| = s_i\}|} = \frac{\binom{N^d-1}{s_i-1}}{\binom{N^d}{s_i}} = \frac{s_i}{N^d}.$$

Theorem 3.10. $\mathbb{E}[|X_1 \cap \ldots \cap X_n|] = \frac{s_1 \ldots s_n}{N^{(n-1)d}}$.

Proof. For each $1 \leq i \leq n$, denote $1_{X_i}(t)$ be the indicator function for the set X_i. Then we have

$$\mathbb{E}[|X_1 \cap \ldots \cap X_n|] = \mathbb{E}\left[\sum_{t \in \mathbb{Z}_N^d} 1_{X_1}(t) \ldots 1_{X_n}(t)\right] = \sum_{t \in \mathbb{Z}_N^d} \mathbb{E}[1_{X_1}(t) \ldots 1_{X_n}(t)] = \sum_{t \in \mathbb{Z}_N^d} \prod_{i=1}^n \mathbb{E}[1_{X_i}(t)].$$

$$= \sum_{t \in \mathbb{Z}_N^d} \prod_{i=1}^n p_{X_i}(t) = \sum_{t \in \mathbb{Z}_N^d} \frac{s_1 \ldots s_n}{N^{nd}} = \frac{s_1 \ldots s_n}{N^{(n-1)d}}.$$

By Markov's inequality, we have

$$\mathbb{P}\left(|X_1 \cap X_2 \cap \ldots \cap X_n| < \frac{N^d}{2\text{spt}(f)}\right) \geq 1 - \frac{\mathbb{E}[|X_1 \cap \ldots \cap X_n|]}{\frac{N^d}{2|\text{spt}(f)|}}$$

$$= 1 - \frac{\frac{s_1 \ldots s_n}{N^{(n-1)d}}}{\frac{N^d}{2|\text{spt}(f)|}} = 1 - \frac{2 s_1 s_2 \ldots s_n |\text{spt}(f)|}{N^{nd}}.$$

Note that $s_i \leq N^d$ so the more signals we send, the higher probability that the size of the intersection does not exceed $\frac{N^d}{2|\text{spt}(f)|}$. However, a drawback is that the verification time will be long, namely $O(min(s_1, \ldots, s_n) \log n)$ via divide and conquer.

Now, we want to use the bilinear restriction theorem for finding the bounds on the size of the supports.

Definition 3.11. Let $X_1, \ldots, X_n$ be sets and t be a number. We define

$$\alpha_{t,n}(X_1, \ldots, X_n) = \left|\left\{(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n : \sum_{i=1}^n x_i = t\right\}\right|.$$

We have the following result.

Theorem 3.12 (Uncertainty principle for bilinear binary signals)**.** *Let $f_1, f_2 : \mathbb{Z}_N^d \to \{0, 1\}$ be two binary signals, where f_i is supported on E_i, $\widehat{f_i}$ is supported on S_i for $i = 1, 2$. Then if $\alpha_2 = \max_{t,2}(\alpha_{t,2}(S_1, S_2))$,*

$$\alpha_2 |E_1||E_2||E_1 \cap E_2|^{-1} \geq N^d.$$

Proof. By the Bilinear Restriction theorem,

$$
\begin{aligned}
&\|f_1f_2\|_{L^2(\mu)} \le \|1_{S_1} * 1_{S_2}\|_\infty^{\frac{1}{2}} \|f_1\|_{L^2(\mu)}\|f_2\|_{L^2(\mu)} \\
\iff& \left(\frac{1}{N^d}|E_1\cap E_2|\right)^{\frac{1}{2}} \le \max_t \alpha_{t,2}(S_1,S_2)^{\frac{1}{2}} \left(\frac{1}{N^d}|E_1|\right)^{\frac{1}{2}} \left(\frac{1}{N^d}|E_2|\right)^{\frac{1}{2}} \\
\iff& N^d|E_1\cap E_2| \le \alpha_2|E_1||E_2| \\
\iff& \alpha_2|E_1||E_2||E_1\cap E_2|^{-1} \ge N^d.
\end{aligned}
$$

One can ask whether this result is strict or not. Let $f_1 = 1_{A_1}, f_2 = 1_{A_2}$ where A_1, A_2 are lines in $\mathbb{Z}_N^2$. Let $A_1 = \{(x, a_1x+b_1) : x \in \mathbb{Z}_p\}, A_2 = \{(x, a_2x+b_2) : x \in \mathbb{Z}_N\}$, where $a_1 \ne a_2 \in \mathbb{Z}_N, \gcd(a_1,N) = \gcd(a_2,N) = 1$ and $b_1, b_2 \in \mathbb{Z}_N$. Then f_i is supported in A_i which has size p so $E_i = A_i$ for $i = 1,2$ and we get $|E_1\cap E_2| = 1$. Now

$$
\begin{aligned}
\widehat{f_1}(m) &= \sum_{x\in\mathbb{Z}_N^2}\chi(-x\cdot m)f(x) = \sum_{\alpha\in\mathbb{Z}_N}\chi(-(\alpha, a_1\alpha+b_1)\cdot m) = \sum_{\alpha\in\mathbb{Z}_N}\chi(-\alpha m_1 - (a_1\alpha+b_1)m_2) \\
&= \sum_{\alpha\in\mathbb{Z}_N}\chi((-m_1-a_1m_2)\alpha - b_1m_2) = \chi(-b_1m_2)\sum_{\alpha\in\mathbb{Z}_N}\chi((-m_1-a_1m_2)\alpha).
\end{aligned}
$$

Note that $\widehat{f_1}(m) \ne 0 \iff -m_1 - a_1m_2 = 0$ so $S_1 = \{(m_1,m_2)\in\mathbb{Z}_N^2 : m_1 + a_1m_2 = 0\}$, which is the line through the origin with slope $\frac{1}{a_1}$. Similarly, S_2 is the line through the origin with slope $\frac{1}{a_2}$. Hence

$$
\alpha_2 = \max_t |S_1 \cap (t - S_2)| = 1
$$

so $\alpha_2|E_1||E_2||E_1\cap E_2|^{-1} = 1\cdot N\cdot N\cdot\frac{1}{1} = N^2$. Hence, this inequality is strict in $\mathbb{Z}_N^2$.

A similar result can be generated in $\mathbb{Z}_N^d$. Let l_1, l_2 be lines in $\mathbb{Z}_N^d$ given by

$$
\begin{cases}
L_1 = \{(a_1,\dots,a_d) + \alpha(b_1,\dots,b_d) : \alpha\in\mathbb{Z}_N\} \\
L_2 = \{(a_1',\dots,a_d') + \alpha(b_1',\dots,b_d') : \alpha\in\mathbb{Z}_N\}
\end{cases}
$$

where we choose the constants such that L_1 intersects L_2. Then similar as above

$$
\widehat{f_1}(m) = \sum_{\alpha\in\mathbb{Z}_N}\chi\left(-\sum_{i=1}^d m_ia_i - \alpha\sum_{i=1}^d m_ib_i\right) = \chi\left(-\sum_{i=1}^d m_ia_i\right)\sum_{\alpha\in\mathbb{Z}_n}\chi\left(-\alpha\sum_{i=1}^d m_ib_i\right).
$$

In order to have $\widehat{f_1}(m) \ne 0$, we need $\sum_{i=1}^{d-1} m_ib_i = 0$ or m is on a $d-1$ hyperplane passing through the origin so the support of $\widehat{f_1}$ is S_1, where

$$
S_1 = \{m\in\mathbb{Z}_d^N : m\cdot b = 0\}, b = (b_1,\dots,b_d).
$$

Similarly, the support of $\widehat{f_2}$ is S_2, where

$$
S_2 = \{m\in\mathbb{Z}_d^N : m\cdot b' = 0\}, b' = (b_1',\dots,b_d').
$$

By choosing b and b' carefully, we can make $\alpha_2 = N^{d-2}$, which is the maximum elements in the intersection of two planes. Hence

$$\alpha_2|E_1||E_2||E_1 \cap E_2|^{-1} = N^{d-2} \cdot N \cdot N \cdot 1 = N^d.$$

Hence, the inequality in Theorem 3.11 is strict.

Now, we want to deduce a version of Theorem 3.11 for a larger family of signals.

Definition 3.13 ($S_{a,b}$)**.** Let $a < b$ be any positive real numbers. The class $\mathcal{S}_{a,b}$ is defined by

$$\mathcal{S}_{a,b} = \{f : \mathbb{Z}_N^d \to \mathbb{C} : a \le |f(x)| \le b \ \forall x \in \mathbb{Z}_N^d, f(x) \neq 0\}.$$

Theorem 3.14 (Uncertainty principle for $\mathcal{S}_{a,b}$ signals)**.** *Let $f_1 \in \mathcal{S}_{a,b}, f_2 \in \mathcal{S}_{c,d}$, where f_i is supported on E_i, $\widehat{f_i}$ is supported on S_i for $i = 1, 2$. Then if $\alpha_2 = \max_t(\alpha_{t,2}(S_1, S_2))$,*

$$\frac{\alpha_2 b^2 d^2 |E_1||E_2|}{a^2 c^2 |E_1 \cap E_2|} \ge N^d.$$

Proof. Again, by the Bilinear Restriction Theorem

$$`\|f_1 f_2\|_{L^2(\mu)} \le \|1_{S_1} * 1_{S_2}\|_\infty^{\frac{1}{2}} \|f_1\|_{L^2(\mu)} \|f_2\|_{L^2(\mu)}$$

$$\iff \left(\frac{1}{N^d}\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2\right)^{\frac{1}{2}} \le \max_t \alpha_{t,2}(S_1,S_2)^{\frac{1}{2}} \left(\frac{1}{N^d}\sum_{x\in E_1}|f_1(x)|^2\right)^{\frac{1}{2}} \left(\frac{1}{N^d}\sum_{x\in E_2}|f_2(x)|^2\right)^{\frac{1}{2}}$$

$$\iff \sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2 \le \frac{\max_t \alpha_{t,2}(S_1,S_2)}{N^d}\left(\sum_{x\in E_1}|f_1(x)|^2\right)\left(\sum_{x\in E_2}|f_2(x)|^2\right)$$

$$\iff \frac{\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2}{\left(\sum_{x\in E_1}|f_1(x)|^2\right)\left(\sum_{x\in E_2}|f_2(x)|^2\right)} \le \frac{\max_t \alpha_{t,2}(S_1,S_2)}{N^d}$$

$$\iff \frac{\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2}{\sum_{x\in E_1, x_2\in E_2}|f_1(x_1)|^2|f_2(x_2)|^2} \le \frac{\max_t \alpha_{t,2}(S_1,S_2)}{N^d}.$$

Now, as $a^2 \le |f_1(x)|^2 \le b^2$ and $c^2 \le |f_2(x)|^2 \le d^2$, we have

$$\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2 = \sum_{x\in E_1\cap E_2}|f_1(x)|^2|f_2(x)|^2 \ge |E_1\cap E_2|c^2a^2$$

and

$$\sum_{x\in E_1, x_2\in E_2}|f_1(x_1)|^2|f_2(x_2)|^2 \le |E_1||E_2|b^2d^2.$$

Hence

$$\frac{|E_1\cap E_2|a^2c^2}{|E_1||E_2|b^2d^2} \le \frac{\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2}{\sum_{x\in E_1, x_2\in E_2}|f_1(x_1)|^2|f_2(x_2)|^2} \le \frac{\max_t \alpha_{t,2}(S_1,S_2)}{N^d} \iff \frac{\alpha_2 b^2 d^2 |E_1||E_2|}{a^2c^2|E_1\cap E_2|} \ge N^d$$

One can question that if a similar result can hold for the class of all signals. In other words, is there a constant $C > 0$ such that for any signals f_1, f_2, if f_i is supported on E_i, $\widehat{f_i}$ is supported on S_i, for $i = 1, 2$, we get

$$C\alpha|E_1||E_2||E_1 \cap E_2|^{-1} \ge N^d.$$

However, this is not the case. With the same proof as above, let $a, c \to \infty$, then $\frac{\sum_{x\in E_1\cap E_2}|f_1(x)f_2(x)|^2}{\sum_{x\in E_1,x_2\in E_2}|f_1(x_1)|^2|f_2(x_2)|^2}$ can get arbitrary small so we cannot bound it by any constant C.

Theorem 3.15 (Uncertainty principle for multilinear binary signals)**.** *Let $f_1, \dots, f_n : \mathbb{Z}_N^d \to \{0,1\}$ be two binary signals, where f_i is supported on E_i, $\widehat{f_i}$ is supported on S_i for $i = 1, \dots, n$. Then if $\alpha_n = \max_t(\alpha_{t,n}(S_1, \dots, S_n))$,*

$$\alpha_n|E_1|\dots|E_n||E_1\cap\dots\cap E_n|^{-1} \geq N^{(n-1)d}.$$

Proof. By the Multilinear Restriction theorem,

$$\begin{aligned}
&\|f_1f_2\dots f_n\|_{L^2(\mu)} \leq \|1_{S_1} * \dots * 1_{S_n}\|_\infty^{\frac{1}{2}}\|f_1\|_{L^2(\mu)}\dots\|f_n\|_{L^2(\mu)}\\
\iff &\left(\frac{1}{N^d}|E_1\cap\dots\cap E_n|\right)^{\frac{1}{2}} \leq \max_t \alpha_{t,n}(S_1,\dots,S_n)^{\frac{1}{2}}\prod_{i=1}^n\left(\frac{1}{N^d}|E_i|\right)^{\frac{1}{2}}\\
\iff &N^{(n-1)d}|E_1\cap\dots\cap E_n| \leq \alpha_n\prod_{i=1}^n|E_i|\\
\iff &\alpha_n|E_1\cap\dots\cap E_n|^{-1}\prod_{i=1}^n|E_i| \geq N^{(n-1)d}.
\end{aligned}$$

Theorem 3.16 (Uncertainty principle for multiple $\mathcal{S}_{a,b}$ signals)**.** *Let $f_i \in \mathcal{S}_{a_i,b_i}$, where f_i is supported on E_i, $\widehat{f_i}$ is supported on S_i for $i = 1, \dots, n$. Then if $\alpha = \max_t(\alpha_t(S_1, \dots, S_n))$,*

$$\frac{\alpha\prod_{i=1}^n b_i^2|E_i|}{\prod_{i=1}^n a_i^2|E_1\cap\dots\cap E_n|} \geq N^{(n-1)d}.$$

Proof. A similar proof follows from Theorem 3.14.

We will also provide Talagrand's bound that will be useful in the next discussion.

Theorem 3.17 (Talagrand, Bourgain [6])**.** *Suppose $h : \mathbb{Z}_N^d \to \mathbb{C}$. Then there exists a $\delta > 0$ such that for a generic set $S, |S| \leq \delta N^d$, we have*

$$\|h\|_{L^2(\mu)} \leq K\sqrt{\log(N)\log(\log(N))}\|h\|_{L^1(\mu)}$$

where $\widehat{h}$ is supported in S.

We will omit the proof. To see how this applied into signal recovery, notice that if we send $\mathrm{Chan}_n(f)$ and let $\widehat{f_i} = \widehat{f}|_{\mathbb{Z}_N^d\setminus S_i}$ where $\widehat{f_i}$ are lost on S_i for any $1 \leq i \leq n$. We see that if E_i is the support of f_i, then $E_1 = E_2 = \dots = E_n = E$. We have

$$\alpha_n|E|^{n-1} \geq N^{(n-1)d} \iff |E| \geq \frac{N^d}{\alpha_n^{\frac{1}{n-1}}}$$

up to a constant.

We will analyze α_n. Note that for any t,

$$\alpha_{t,n}(S_1,\ldots,S_n) = \left|(t_1,\ldots,t_n) \in S_1 \times \ldots \times S_n : \sum t_i = t\right|.$$

Fixing a $t \in \mathbb{Z}_N^d$, by determining $t_1,\ldots,t_{n-1}$, we can get the value of t_n uniquely. Hence there are $N^{d(n-1)}$ solutions to the equation $t_1 + \ldots + t_n = t$ where $t_i \in \mathbb{Z}_N^d$. Hence

$$\mathbb{E}[\alpha_{t,n}(S_1,\ldots,S_n)] = \mathbb{E}\left[\sum_{t_i \in \mathbb{Z}_N^d : \sum_{t_i} = t} 1_{S_1}(t_1)\ldots 1_{S_n}(t_n)\right] = \sum_{t_i \in \mathbb{Z}_N^d, \sum t_i = t} \mathbb{E}[1_{S_1}(t_1)]\ldots \mathbb{E}[1_{S_n}(t_n)]$$
$$= \sum_{t_i \in \mathbb{Z}_N^d, \sum t_i = t} \frac{|S_1|\ldots|S_n|}{N^{dn}} = N^{-d}\prod_{i=1}^n |S_i|.$$

Then we have

$$\mathbb{E}[\alpha_n] = \mathbb{E}[\max_t \alpha_{t,n}(S_1,\ldots,S_n)] \geq \max_t \mathbb{E}[\alpha_{t,n}(S_1,\ldots,S_n)] = N^{-d}\prod_{i=1}^n |S_i|.$$

If we assume that each S_i satisfies the Talagrand bound, then

$$\|f_1\ldots f_n\|_{L^2(\mu)} \leq \alpha_n^{\frac{1}{2}} \prod_{i=1}^n \|f_i\|_2 \leq \alpha_n^{\frac{1}{2}}(K\sqrt{\log(\log(N))\log(N)})^n \|f_i\|_{L^1(\mu)}.$$

From above, we want $f_1,\ldots,f_n$ to be equal so as $E_1 = \ldots = E_n = E$, we get

$$N^{-\frac{d}{2}}\left(\sum_{x\in E}|f_1(x)\ldots f_n(x)|^2\right)^{\frac{1}{2}} \leq \alpha_n^{\frac{1}{2}}(K\sqrt{\log(\log(N))\log(N)})^n N^{-dn}\prod_{i=1}^n\left(\sum_{x\in E}|f_i(x)|\right).$$

Suppose $f \in \mathcal{S}_{a,b}$ for some $a < b \in \mathbb{R}$. Then from above, we get

$$N^{-\frac{d}{2}}|E|^{\frac{1}{2}} \lesssim_{a,b} \alpha_n^{\frac{1}{2}}(K\sqrt{\log(\log(N))\log(N)})^n N^{-dn}|E|^n.$$

This is equivalent to $N^{-d}|E| \lesssim_{a,b} \alpha_n(K^2\log(\log(N))\log(N))^n N^{-2dn}|E|^{2n}$ or we get

$$|E| \gtrsim_{a,b} \frac{N^d}{\alpha_n^{\frac{1}{2n-1}}(K^2\log(\log(N))\log(N))^{\frac{n}{2n-1}}}.$$

With the approximation of $\alpha_n \approx N^{-d}\prod_{i=1}^n|S_i| \approx \delta^n N^{(n-1)d}$, then the denominator is approximately $\delta^{\frac{n}{2n-1}}N^{\frac{d(n-1)}{2n-1}}(K^2\log(\log(N))\log(N))^{\frac{n}{2n-1}}$, which decreases as n increases.

Now, suppose we execute Logan's method on the problem $g = \arg\min_u \|u\|_1$ where $\widehat{u}(x) = \widehat{f_i}(x)$ away from S_i and $f, g \in \mathcal{S}_{a,b}$ for some a, b. Let $g = f + h$. Then we see that

$$\|g\|_1 \geq \|f\|_1 + (\|h\|_{L^1(E^c)} - \|h\|_{L^1(E)}).$$

Moreover, note that

$$|h| = |f - g| \leq |f| + |g| = 2b$$

so we have $h \in \mathcal{S}_{0,2b}$. From the multilinear restricition, we have

$$\|h^n\|_{L^2(\mu)} \leq \alpha_n^{\frac{1}{2}} \prod_{i=1}^{n} \|h\|_{L^2(\mu)}$$

or we get $\|h\|_{L^{2n}(\mu)} \leq \alpha_n^{\frac{1}{2n}} \|h\|_{L^2(\mu)}$ By Talagrand's bound and Holder's Inequality, we have

$$\begin{aligned}\|h\|_{L^1(E)} &= \sum_{x\in E} |h(x)| \leq \left(\sum_{x\in E} |h(x)|^{2n}\right)^{\frac{1}{2n}} |E|^{\frac{2n-1}{2n}} = N^{\frac{d}{2n}} \|h\|_{L^{2n}(\mu)} |E|^{\frac{2n-1}{2n}} \\ &\leq \alpha_n^{\frac{1}{2n}} N^{\frac{d}{2n}} |E|^{\frac{2n-1}{2n}} \|h\|_{L^2(\mu)} \leq \alpha_n^{\frac{1}{2n}} N^{\frac{d(1-2n)}{2n}} |E|^{\frac{2n-1}{2n}} (K \log(\log(N)) \log(N)) \|h\|_{L^1}.\end{aligned}$$

Then we have

$$\alpha_n^{\frac{1}{2n}} N^{\frac{d(1-2n)}{2n}} |E|^{\frac{2n-1}{2n}} (K \log(\log(N)) \log(N)) < \frac{1}{2} \iff |E| < \frac{N^d}{\alpha_n^{\frac{1}{2n-1}} (4K^2 \log(\log(N)) \log(N))^{\frac{n}{2n-1}}}.$$

Hence, if $|E| < \frac{N^d}{\alpha_n^{\frac{1}{2n-1}} (4K^2 \log(\log(N)) \log(N))^{\frac{n}{2n-1}}}$, then we have $\|h\|_{L^1(E^c)} - \|h\|_{L^1(E)} > 0$, contradicting the condition of minimum of g. As n increases, the bound $\frac{N^d}{\alpha_n^{\frac{1}{2n-1}} (4K^2 \log(\log(N)) \log(N))^{\frac{n}{2n-1}}}$ become larger so we can recover f more easily when we send a higher channel with Logan's method and Talagrand's bound.

We see that indeed sending multiple signals can improve the Uncertainty Principle, so we want to develop an algorithm for that. The first idea is to use the merge function.

Definition 3.18 ($Merge(f_1, \ldots, f_n)$)**.** For signals $f_1, \ldots, f_n : \mathbb{Z}_N^d \to \mathbb{C}$, where the values of f_i are lost on S_i, we define the signal

$$Merge(f_1, \ldots, f_n)(x) = \begin{cases} f_i(x) & \text{if } x \notin S_i \text{ for some } 1 \leq i \leq n \\ \text{undefined} & \text{otherwise} \end{cases}.$$

Then consider the following algorithm.

Definition 3.19 (Generalization of L_2-minimization)**.** Let $g = \arg\min \|u\|_2$, where $\widehat{u}(m) = Merge(\widehat{f_1}, \ldots, \widehat{f_n})(m)$ for any $m \notin S_1 \cap \ldots \cap S_n$.

As described above, if $|S_1 \cap \ldots \cap S_n||E| \leq \frac{N^d}{2}$, then we can recover g uniquely. However, the time complexity for $Merge$ can take exponential time.

4. Bourgain's Λ_q problem

We state the original Bourgain Λ_q problem.

Theorem 4.1 (Bourgain [1])**.** *Let $\Psi = (\psi_1, \dots, \psi_n)$ denote a sequence of n mutually orthogonal functions, with $\|\psi_i\|_{L^\infty(G)} \leq 1$. There exists a subset S of $\{1, \dots, n\}, |S| > n^{\frac{2}{q}}$ such that*

$$\left\|\sum_{i\in S} a_i\psi_i\right\|_{L^q(G)} \leq C(q)\left(\sum_{i\in S}|a_i|^2\right)^{\frac{1}{2}}.$$

for all finite sequence $\{a_i\}$. The estimate holds for a generic set of size $\lceil N^{\frac{2}{q}}\rceil$ with probability $1 - o_N(1)$.

Let $G = \mathbb{Z}_N^d$ and consider $\Gamma = \{\gamma(x \cdot m) : m \in \mathbb{Z}_N^d\}$. Applying Bourgain's result, if we consider $n = N^d$, $|S| = \lceil N^{\frac{2d}{q}}\rceil$, and $\{a_i\} = \{N^{-\frac{d}{2}}\widehat{f}(x)\}_{x\in S}$, where $\widehat{f}$ is supported in S, then almost surely

$$\left\|\sum_{m\in S} N^{-\frac{d}{2}}\widehat{f}(m)\chi(m\cdot x)\right\|_{L^q(G)} \leq C(q)\left(\frac{1}{N^d}\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}} = C(q)\left(\frac{1}{N^d}\sum_{x}|f(x)|^2\right)^{\frac{1}{2}}$$

$$\iff \|f(x)\|_{L^q(G)} \leq C(q)\left(\frac{1}{N^d}\sum_{x}|f(x)|^2\right)^{\frac{1}{2}}.$$

Hence, we get the variation of Bourgain for discrete analouge.

Theorem 4.2 (Variation of Bourgain in Discrete settings)**.** *If $\widehat{f}$ is supported in S and $|S| = \lceil N^{\frac{2d}{q}}\rceil$, then*

$$\|f\|_{L^q(\mu)} \leq C(q)\|f\|_{L^2(\mu)}$$

where $C(q)$ is a constant independent of N.

Lemma 4.3. *Proof the theorem is correct for $q = 4$. That is, if $|S| = \lceil N^{\frac{d}{2}}\rceil$, then*

$$\left(\frac{1}{N^d}\sum_{x}|f(x)|^4\right)^{\frac{1}{4}} \leq C(q)\left(\frac{1}{N^d}\sum_{x}|f(x)|^2\right)^{\frac{1}{2}}.$$

Proof. We have

$$\begin{aligned}
LHS = \left(\frac{1}{N^d}\sum_{x}(|f(x)|^2)^2\right)^{\frac{1}{4}} &= \left(\frac{1}{N^d}\sum_{x}(f(x)\overline{f(x)})^2\right)^{\frac{1}{4}}\\
&= \left(\frac{1}{N^d}\sum_{x}\left(\left(N^{-\frac{d}{2}}\sum_{m\in S}\chi(x\cdot m)\widehat{f}(m)\right)\left(N^{-\frac{d}{2}}\sum_{m\in S}\chi(-x\cdot m)\overline{\widehat{f}(m)}\right)\right)^2\right)^{\frac{1}{4}}\\
&= \left(\frac{1}{N^{3d}}\sum_{x}\left(\sum_{m_1,m_2\in S}\chi(x\cdot(m_1-m_2)\widehat{f}(m_1)\overline{\widehat{f}(m_2)}\right)^2\right)^{\frac{1}{4}}.
\end{aligned}$$

By expanding out again, we have

$$LHS = \left(\frac{1}{N^{3d}}\sum_x \left(\sum_{m_1,m_2\in S}\chi(x\cdot(m_1-m_2)\widehat{f}(m_1)\overline{\widehat{f}(m_2)}\right)\left(\sum_{m_1',m_2'\in S}\chi(x\cdot(m_1'-m_2')\widehat{f}(m_1')\overline{\widehat{f}(m_2')}\right)\right)^{\frac{1}{4}}$$

$$= \left(\frac{1}{N^{3d}}\sum_x \sum_{m_1,m_2,m_1',m_2'\in S}\chi(x\cdot(m_1-m_2+m_1'-m_2'))\widehat{f}(m_1)\widehat{f}(m_1')\overline{\widehat{f}(m_2)\widehat{f}(m_2')}\right)^{\frac{1}{4}}.$$

$$= \left(\frac{1}{N^{3d}}\sum_{m_1,m_2,m_1'm_2'\in S}\widehat{f}(m_1)\widehat{f}(m_1')\overline{\widehat{f}(m_2)\widehat{f}(m_2')}\sum_x\chi(x\cdot(m_1-m_2+m_1'-m_2'))\right)^{\frac{1}{4}}.$$

Again, by Gauss sum,

$$LHS = N^{-\frac{d}{2}}\left(\sum_{\substack{m_1,m_2,m_1',m_2'\in S\\ m_1+m_1'=m_2+m_2'}}\widehat{f}(m_1)\widehat{f}(m_1')\overline{\widehat{f}(m_2)\widehat{f}(m_2')}\right)^{\frac{1}{4}}.$$

For the right hand side, by Plancherel formula

$$RHS = \left(\frac{1}{N^d}\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}} = N^{-\frac{d}{2}}\left(\sum_{m\in S}\widehat{f}(m)\overline{\widehat{f}(m)}\right)^{\frac{1}{2}}.$$

Hence, the inequality is equivalent to

$$\sum_{\substack{m_1,m_2,m_1',m_2'\in S\\ m_1+m_1'=m_2+m_2'}}\widehat{f}(m_1)\widehat{f}(m_1')\overline{\widehat{f}(m_2)\widehat{f}(m_2')} \le C(q)^2\left(\sum_{m\in S}\widehat{f}(m)\overline{\widehat{f}(m)}\right)^2$$

$$= C(q)^2\sum_{m_1,m_2\in S}\widehat{f}(m_1)\widehat{f}(m_2)\overline{\widehat{f}(m_1)\widehat{f}(m_2)}.$$

If we consider $\widehat{f}=1$, then the inequality is equivalent to

$$|\{(m_1,m_2,m_3,m_4)\in S^4 : m_1+m_2=m_3+m_4\}| \le C(q)^2|S|^2.$$

Since $|S| > N^{\frac{d}{2}}$, we see that the left hand side is bounded by $(5+o(1))|S|^2$ so taking $C(q)=\sqrt{5+o(1)}$ will satisfy.

In other cases, we take $C(q)=\sqrt{5+o(1)}\frac{\max\widehat{f}}{\min\widehat{f}}$ and we get the results.

To see how Bourgain's result can be applied in signal recovery, we consider the following theorem proved by Iosevich and Mayeli [3].

Theorem 4.4 (Iosevich, Mayeli [3])**.** *Suppose $f:\mathbb{Z}_N^d\to\mathbb{C}$ is a signal supported in E. Suppose that $\{\widehat{f}(m)\}_{m\in S}$ are unobserved where S satisfies the Λ_q inequality with constant $C(q)$, then f can be recovered exactly if*

$$|E| < \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}.$$

We want to generalize the idea for multiple signals. Let $f_1, \ldots, f_n : \mathbb{Z}_N^d \to \mathbb{C}$ be functions. Note that

$$\widehat{f_1 \ldots f_n} = N^{\frac{(n-1)d}{2}} (\widehat{f_1} * \ldots * \widehat{f_n})$$

so $\widehat{f_1 \ldots f_n}(t) = \sum_{\sum x_i = t} \widehat{f_1}(x_1) \ldots \widehat{f_n}(x_n)$. If $t \notin S_1 + \ldots + S_n$, then $\widehat{f_1 \ldots f_n}(t) = 0$ so if $\widehat{f_1 \ldots f_n}(t) \neq 0$, we must have $t \in S_1 + \ldots + S_n$ and we get

$$\operatorname{spt}(\widehat{f_1 \ldots f_n}) \subseteq S_1 + \ldots + S_n.$$

If we have $|\operatorname{spt}(\widehat{f_1 \ldots f_n})| = \lceil N^{\frac{2d}{q}} \rceil$, then by Bourgain's result and Multilinear restriction

$$\|f_1 \ldots f_n\|_{L^q(\mu)} \leq C(q) \|f_1 \ldots f_n\|_{L^2(\mu)} \leq C(q) \|1_{S_1} * \ldots * 1_{S_n}\|_\infty^{\frac{1}{2}} \prod_{i=1}^{k} \|f_i\|_{L^2(\mu)}.$$

We have $|S_1 + \ldots + S_n| \geq \lceil N^{\frac{2d}{q}} \rceil$ and with high probability, by Pigeonhole Principle

$$\|1_{S_1} * \ldots * 1_{S_n}\|_\infty^{\frac{1}{2}} = \max_t \alpha_{t,n}(S_1, \ldots, S_n)^{\frac{1}{2}} \approx \left(\frac{\prod_{i=1}^n |S_i|}{|S_1 + \ldots + S_n|} \right)^{\frac{1}{2}} \leq \left(\frac{\prod_{i=1}^n |S_i|}{\lceil N^{\frac{2d}{q}} \rceil} \right)^{\frac{1}{2}}$$

so if we choose S_i with size $N^{\frac{2d}{qn}}$ for any i, then $\left(\frac{\prod_{i=1}^n |S_i|}{\lceil N^{\frac{2d}{q}} \rceil} \right)^{\frac{1}{2}} = O(1)$. Note that with $n = 1$, it is the original Bourgain's theorem.

Now, we want to find the expected value of $|S_1 + \ldots + S_n|$. Note that for any $x \in \mathbb{Z}_N^d, \mathbb{P}(x \in S_i) = \frac{|S_i|}{N^d}$. Hence

$$\mathbb{E}[|S_1 + \ldots + S_n|] = \sum_{x \in \mathbb{Z}_N^d} \mathbb{P}(x \in S_1 + \ldots + S_n).$$

For each $1 \leq i \leq n$, let $u_i = (u_{i1}, \ldots, u_{in}) \in \mathbb{Z}_N^d$ such that $u_1 + \ldots + u_n = x$. Consider the whole space $\mathbb{Z}_N^d$, this equation has $N^{d(n-1)}$ solutions, called it $Sol(x)$. So if $x \notin S_1 + \ldots + S_n$, then for any of the $N^{d(n-1)}$ tuples $(u_1, \ldots, u_n) \in Sol(x)$, there exists some i such that $u_i \notin S_i$. Hence

$$\mathbb{P}(x \notin S_1 + \ldots + S_n) = \mathbb{P}(\forall (u_1, \ldots, u_n) \in Sol(x), \exists i : u_i \notin S_i).$$

Numbering all elements in $Sol(x)$. Let E_j be the event that there is some $1 \leq i \leq n$ such that $u_{ji} \notin S_i$ where $(u_{j1}, \ldots, u_{jn})$ is the jth element in $Sol(x)$. Then the events $\{E_j\}_{j=1}^{N^{d(n-1)}}$ are pairwise independence so $\mathbb{P}(x \notin S_1 + \ldots + S_n) = \mathbb{P}(E_1)^{N^{d(n-1)}}$ since any E_j occurs with the same probability. We also have

$$\mathbb{P}(E_1) = 1 - \mathbb{P}(u_{i1} \in S_i \ \forall 1 \leq i \leq n) = 1 - \frac{|S_1| \ldots |S_n|}{N^{nd}}$$

so we must have

$$\mathbb{E}[|S_1 + \ldots + S_n|] = N^d \left(1 - \left(1 - \frac{|S_1| \ldots |S_n|}{N^{nd}} \right)^{N^{d(n-1)}} \right).$$

From above, if we take each S_i has size approximately $N^{\frac{2d}{nq}}$, then $\frac{|S_1|\dots|S_n|}{N^{nd}} = N^{\frac{2d}{q}-nd}$ so

$$\mathbb{E}[|S_1+\ldots+S_n|] = N^d\left(1-\left(1-N^{d(\frac{2}{q}-n)}\right)^{N^{d(n-1)}}\right)$$

and as $N^{d(\frac{2}{q}-n)}N^{d(n-1)} = N^{d(\frac{2}{q}-1)} \ll 1$ if we take $q \geq 3$, we see that since $(1-x)^n \approx 1-nx$ when $n|x| \ll 1$,

$$\left(1-N^{d(\frac{2}{q}-n)}\right)^{N^{d(n-1)}} \approx 1-N^{d(n-1)}N^{d(\frac{2}{q}-n)} = 1-N^{d(\frac{2}{q}-1)}$$

and we get

$$\mathbb{E}[|S_1+\ldots+S_n|] \approx N^d(1-(1-N^{d(\frac{2}{q}-1)})) = N^dN^{d(\frac{2}{q}-1)} = N^{\frac{2d}{q}}.$$

Hence, as $|S_1+\ldots+S_n| \leq |S_1|\ldots|S_n| = N^{\frac{2d}{q}}$, then with high probability, $|S_1+\ldots+S_n| = N^{\frac{2d}{q}} = |S_1|\ldots|S_n|$ so $\|1_{S_1}*\ldots*1_{S_n}\|_\infty^{\frac{1}{2}} = 1$. We get the following result.

Theorem 4.5 (Generalization of Bourgain's Theorem)**.** *If for any $1 \leq i \leq n$, $\widehat{f_i}$ is supported in S_i with $|S_i| = \lceil N^{\frac{2d}{nq}}\rceil$ and also $\widehat{f_1\ldots f_n}$ is supported in S with $|S| = \lceil N^{\frac{2d}{q}}\rceil$, then with high probability*

$$\|f_1\ldots f_n\|_{L^q(\mu)} \leq C(q)\prod_{i=1}^n \|f_i\|_{L^2(\mu)}$$

where $C(q)$ is a constant independent of N.

Acknowledgement

First of all, I would like to thank Professor Alex Iosevich, Professor Dan-Andrei Geba, and Professor Jonathan Pakianathan for their comments and suggestions on the content of the thesis. I would also like to thank the Mathematics Department for their support with researches and classes, as well as my family and friends for their wonderful advices during my undergraduate years. I would not be success without their help.

References

[1] J. Bourgain. Bounded orthogonal systems and the $\lambda(p)$-set problem, 1989. 18
[2] David L. Donoho and Philip B. Stark. Uncertainty principles and signal recovery. SIAM Journal on Applied Mathematics, 49(3):906–931, 1989. 1
[3] A. Iosevich and A. Mayeli. Uncertainty principles on finite abelian groups, restriction theory, and applications to sparse signal recovery, 2023. 6, 19
[4] A. Iosevich and A. Mayeli. Uncertainty principles, restriction, bourgain's λ_q theorem, and signal recovery, 2025. 3, 4
[5] Ben Krause. Discrete analogues in harmonic analysis: Bourgain, stein, and beyond, 2022. 7
[6] M. Talagrand. Selecting a proportion of characters, 1998. 15

Department of Mathematics, University of Rochester, Rochester, NY
Email address: tle18@u.rochester.edu