Experiments with the Bourgain Constant and Its Applications to Signal Recovery

Terrence Wong

April 24, 2025

1 Introduction

The Bourgain constant comes from a result from Bourgain's celebrated 1989 paper [1] stated as follows

Theorem 1.1. *Let G be a locally compact abelian group. Let $\Psi = (\psi_1, \dots, \psi_n)$ denote a sequence of n mutually orthogonal functions, with $\|\psi_i\|_{L^\infty(G)} \leq 1$. There exists a subset S of $\{1, 2, \dots, n\}$, $|S| > n^{\frac{2}{q}}$ such that*

$$\left\| \sum_{i \in S} a_i \psi_i \right\|_{L^q(G)} \leq C(q) \left(\sum_{i \in S} |a_i|^2 \right)^{\frac{1}{2}}$$

where the constant $C(q)$ depends only on q and the estimate above holds for a generic set of size $\lceil n^{\frac{2}{q}} \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than x

This constant is very important to the signal recovery problem, and has wide-ranging applications throughout computer science, data science, mathematics, and engineering. However, the value of this constant is currently unknown. In this paper, we perform numerical experiments to determine a rough estimate of the Bourgain constant. We will also implement various signal recovery methods and compare their efficacy to their theoretical connections to the Bourgain constant.

2 Preliminaries

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a given signal (function) where $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. The Fourier transform of f is given as a function $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ such that

$$\hat{f}(m) = N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x)$$

where $\chi(t) = e^{\frac{2\pi it}{N}}$, $t \in \mathbb{Z}_N$ and $m \in \mathbb{Z}_N^d$. The inverse Fourier transform is given as

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m)$$

With this normalization of the Fourier transform, the Plancherel identity is given by

$$\sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 = \sum_{m \in \mathbb{Z}_N^d} |\hat{f}(m)|^2$$

The classical Fourier uncertainty principle relates the support of an arbitrary function with the support of its Fourier transform. It states that for a given nonzero $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$|\mathrm{supp}(f)| \cdot |\mathrm{supp}(\hat{f})| \geq N^d$$

where $|\mathrm{supp}(f)|$ is defined to be the number of nonzero entries in f.

Suppose some of the Fourier coefficients of f are not known. We wish to find conditions under which the original signal can be exactly recovered. More precisely, for an arbitrary subset $S \subset \mathbb{Z}_N^d$, we want to reconstruct f when only the values $\{\hat{f}(m)\}_{m \notin S}$ are known. This problem was extensively studied in a seminal paper by Donoho and Stark [2], where they proved the following result.

Theorem 2.1. *Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a finite signal in $\mathbb{Z}_N$ with length N and N_t nonzero entries. Suppose that the set of unobserved frequencies $\{\hat{f}(m)\}_{m \in \mathbb{Z}_N}$ is of size N_w. Then the signal f can be recovered uniquely from the observed frequencies if*

$$N_t \cdot N_w < \frac{N}{2} \tag{2.1}$$

For a signal to be successfully recovered, it must have some degree of sparsity, and a limited number of frequencies can be absent. Under these conditions, the signal that is recovered is also guaranteed to be unique. To show this, let $f : \mathbb{Z}_N \to \mathbb{C}$ with $f \neq 0$, let $E = \mathrm{supp}(f)$, and let $S \subset \mathbb{Z}_N$ be the set of frequencies on which $\{f(m)\}_{m \in S}$ is unknown. Suppose that the recovery is not unique. Then, there exists a signal g such that $\hat{g}(m) = \hat{f}(m)$ for $m \notin S$, $|\mathrm{supp(g)}| = |\mathrm{supp}(f)|$, and $f \neq g$. Let $h = f - g \neq 0$. By the triangle inequality, $|\mathrm{supp}(h)| \leq |\mathrm{supp}(f)| + |\mathrm{supp}(g)| = 2|\mathrm{supp}(f)|$. At the same time, $\hat{h}(m) = 0$ for $m \notin S$, so $\mathrm{supp}(\hat{h}) \subset S$. By the uncertainty principle, $|\mathrm{supp}(h)| \cdot |\mathrm{supp}(\hat{h})| = 2|\mathrm{supp}(f)| \cdot |S| \geq N$. This forms a contradiction with assumption (2.1). Thus, the recovered signal is unique.

There have been a number of results that improve on the recovery condition (2.1). The ones that we will discuss in this paper build upon a celebrated result by Jean Bourgain [1] as follows.

Theorem 2.2. *Let G be a locally compact abelian group. Let $\Psi = (\psi_1, \dots, \psi_n)$ denote a sequence of n mutually orthogonal functions, with $\|\psi_i\|_{L^\infty(G)} \le 1$. There exists a subset S of $\{1, 2, \dots, n\}$, $|S| > n^{\frac{2}{q}}$ such that*

$$\left\|\sum_{i\in S} a_i\psi_i\right\|_{L^q(G)} \le C(q)\left(\sum_{i\in S}|a_i|^2\right)^{\frac{1}{2}}$$

where the constant $C(q)$ depends only on q and the estimate above holds for a generic set of size $\lceil n^{\frac{2}{q}}\rceil$, where $\lceil x\rceil$ denotes the smallest integer greater than x

We will refer to this constant $C(q)$ as the Bourgain constant, and it is the main focus of our studies in this paper.

Taking $a_x = \hat{f}(x)$ and $\Psi = \{\chi(x\cdot m) : m \in \mathbb{Z}_N^d\}$, multiplying both sides by $N^{-\frac{d}{2}}$, and applying Plancharel's identity to the right hand side leads to the following notable consequence.

Corollary 2.3. *Given $f : \mathbb{Z}_N^d \to \mathbb{C}$, let*

$$\hat{f}(m) = N^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}_N^d} \chi(-x\cdot m)f(x),\ x\cdot m = x_1m_1 + \cdots + x_dm_d$$

where $\chi(t) = e^{\frac{2\pi it}{N}}$. Then for a generic subset Σ of $\mathbb{Z}_N^d$ of size $\lceil N^{\frac{2d}{q}}\rceil$, $q > 2$, if $\hat{f}$ is supported in Σ, we have

$$\|f\|_{L^q(\mu)} \le C(q)\|f\|_{L^2(\mu)} \tag{2.2}$$

where $C(q)$ depends only on q, and here, and throughout,

$$\|f\|_{L^p(\mu)} = \left(\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|f(x)|^p\right)^{\frac{1}{p}}$$

Iosevich and Mayeli [3] used this result and the Donoho-Stark argument to prove the following result.

Theorem 2.4. *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ supported in $E \subset \mathbb{Z}_N^d$. Let r be a frequency bandlimited signal given by*

$$\hat{r}(m) = \begin{cases} \hat{f}(m), & \text{for } m \notin S \\ 0, & \text{otherwise} \end{cases}$$

where S is a subset of $\mathbb{Z}_N^d$ of size $\lceil N^{\frac{2d}{q}}\rceil$, for some $q > 2$, randomly chosen with uniform probability. Then there exists a constant $C(q)$ that depends only on q such that with probability $1 - o(1)$, if

$$|E| < \frac{N^d}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}} \tag{2.3}$$

then f can be reconstructed from r uniquely.

Proof. Let f be a signal supported in E and $\hat{f}$ supported in Σ of size $\lceil N^{\frac{2d}{q}} \rceil$. We see that the left hand side of (2.2) can be rewritten as

$$N^{-\frac{d}{q}}|E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}}$$

while the right hand side can be rewritten as

$$C(q)N^{-\frac{d}{2}}|E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}}$$

By Hölder's Inequality, it can be shown that

$$\left(\frac{1}{|E|} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}} \leq \left(\frac{1}{|E|} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}}$$

Using this fact and rearranging terms gives that

$$|E| \geq \frac{N^d}{(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}$$

Now, let f be supported in E and S with size $\lceil N^{\frac{d}{q}} \rceil$ be the set of frequencies on which $\{f(m)\}_{m \in S}$ is unknown. Suppose there exists g such that $\hat{g}(m) = \hat{f}(m)$ for $m \notin S$, $|\text{supp}(g)| = |\text{supp}(f)|$, and $f \neq g$. Let $h = f - g$. Then $|\text{supp}(\hat{h})| = |S|$, and

$$2|E| \geq |\text{supp}(h)| \geq \frac{N^d}{(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}$$

This contradicts assumption (2.3), so the recovery is unique.

This result gives a much better recovery condition for signals that fit the assumptions, provided that we know what $C(q)$ is. However, it is currently not known how to compute the Bourgain constant. In this paper, we aim to experimentally find a numerical estimation of the Bourgain constant.

We will analyze three algorithms for signal recovery in relation to the Bourgain constant. The first of which the Direct Rounding Algorithm, which is applied specifically to binary signals. Let $E(x)$ be the indicator function of a set $E \subset \mathbb{Z}_N^d$. Suppose that the values of $\hat{E}(m)$ are not known for $m \in S$. Let $r : \mathbb{Z}_N^d \to \mathbb{C}$ be such that

$$r(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d \setminus S} \hat{E}(m) \chi(m \cdot x)$$

thus

$$\hat{r}(m) = \begin{cases} \hat{E}(m) & m \notin S \\ 0 & m \in S \end{cases}$$

We define $G(x)$ such that

$$G(x) = \begin{cases} 1 & |r(x)| \geq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

If $E(x) = G(x)$ for all $x \in \mathbb{Z}_N^d$, then we say that E can be recovered by the DRA.

Theorem 2.5. *Let $E \subset \mathbb{Z}_N^d$ and $S \subset \mathbb{Z}_N^d$. Suppose that the values of $\hat{E}(m)$ are not known for $m \in S$. Then E can be recovered by the Direct Rounding Algorithm if*

$$|E||S| < \frac{N}{2}$$

In other words, if the Donoho-Stark condition (2.1) holds for a binary signal, then the DRA can be used to recover it.

Proof. Let $E \subset \mathbb{Z}_N^d$, $S \subset \mathbb{Z}_N^d$. We can write

$$\begin{aligned} E(x) &= N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{E}(m) \\ &= N^{-\frac{d}{2}} \sum_{m \notin S} \chi(x \cdot m) \hat{E}(m) + N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \hat{E}(m) \\ &= I(x) + II(x) \end{aligned}$$

Using the triangle inequality and definition of the Fourier transform,

$$|II(x)| \leq N^{-\frac{d}{2}} \sum_{m \in S} |\hat{E}(m)| \leq N^{-\frac{d}{2}} \sum_{m \in S} N^{-\frac{d}{2}} \sum_{x \in S} |E(x)| = N^{-d}|E||S|$$

If the above quantity is less than $\frac{1}{2}$, then rounding is able to reconstruct $E(x)$ from $I(x)$.

Another recovery method that we will discuss is the L^1 minimization algorithm [4]. For a signal $f : \mathbb{Z}_N^d \to \mathbb{C}$ with support $E \subset \mathbb{Z}_N^d$ and $\hat{f}(m)$ unknown for $S \subset \mathbb{Z}_N^d$, we choose $g = \underset{u}{\operatorname{argmin}} \|u\|_1$ such that $\hat{u}(m) = \hat{f}(m)$ for $m \notin S$. We say that f can be recovered by L^1 minimization if $g = f$.

Theorem 2.6. *If S is a generic set of size $\lceil N^{\frac{2d}{q}} \rceil$ with $q \geq 3$, then f can be recovered by L^1 minimization if*

$$|E| < \frac{N}{4(C(q))^6} \tag{2.4}$$

where $C(q)$ is the Bourgain constant for q.

Proof. Note that f is in the set of u, so $\|g\|_1 \leq \|f\|_1$. Let $h = f - g$, and assume that $h \neq 0$. We can write

$$\begin{aligned}\|g\|_1 = \|f-h\|_1 &= \|f-h\|_{L^1(E)} + \|f-h\|_{L^1(E^c)} \\ &\geq \|f\|_1 + (\|h\|_{L^1(E^c)} - \|h\|_{L^1(E)})\end{aligned}$$

If $\|h\|_{L^1(E)} < \|h\|_{L^1(E^c)}$, then the term in parentheses above would be greater than 0 and we get a contradiction with the fact that g is a minimizer, thus showing that $h = 0$ and that f was successfully recovered.

Using the Cauchy-Schwarz inequality and representing $h^2 = h^{\frac{1}{2}} h^{\frac{3}{2}}$, we write

$$\|h\|_{L^2(\mu)} \leq \|h\|_{L^1(\mu)}^{\frac{1}{4}} \|h\|_{L^3(\mu)}^{\frac{3}{4}}$$

By the assumption that $q \geq 3$ and that S is generic with size $\lceil N^{\frac{2d}{q}} \rceil$, then using Hölder's Inequality and Theorem 2.2, we can show that

$$\begin{aligned}\|h\|_{L^2(\mu)} &\leq \|h\|_{L^1(\mu)}^{\frac{1}{4}} \|h\|_{L^q(\mu)}^{\frac{3}{4}} \\ &\leq \|h\|_{L^1(\mu)}^{\frac{1}{4}} \|h\|_{L^q(\mu)}^{\frac{3}{4}} \\ &\leq \|h\|_{L^1(\mu)}^{\frac{1}{4}} (C(q))^{\frac{3}{4}} \|h\|_{L^2(\mu)}^{\frac{3}{4}} \\ \|h\|_{L^2(\mu)} &\leq (C(q))^3 \|h\|_{L^1(\mu)}\end{aligned}$$

Using the above result and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|h\|_{L^1(E)} &\leq |E|^{\frac{1}{2}} \|h\|_{L^2(E)} \\ &\leq |E|^{\frac{1}{2}} \|h\|_2 \\ &= |E|^{\frac{1}{2}} N^{\frac{d}{2}} \|h\|_{L^2(\mu)} \\ &\leq |E|^{\frac{1}{2}} N^{\frac{d}{2}} (C(q))^3 \|h\|_{L^1(\mu)} \\ &= |E|^{\frac{1}{2}} N^{-\frac{d}{2}} (C(q))^3 \|h\|_1\end{aligned}$$

Thus if (2.4) holds, then $\|h\|_{L^1(E)} \leq \frac{1}{2}\|h\|_1$ and $\|h\|_{L^1(E)} \leq \|h\|_{L^1(E^c)}$. Therefore, there is a contradiction and $f = g$.

3 Methods

All of the following algorithms were implemented and tested in Python. As we are performing experiments on random signals, our data will have some variance and different trials will have varying values. As such, it is difficult to draw exact

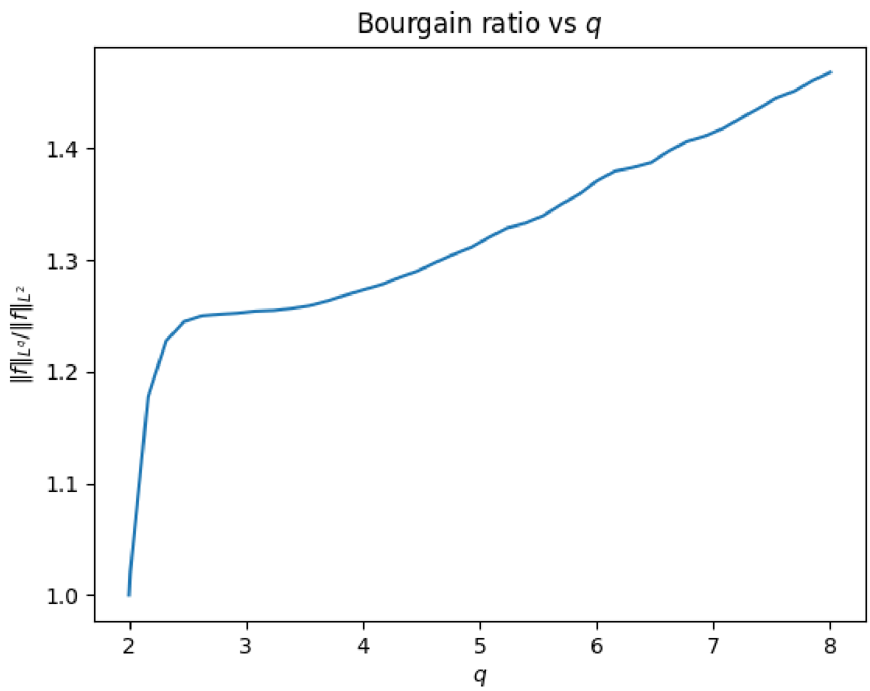

Figure 1: Bourgain ratio with respect to q. q varies from 2 to 8. N is fixed at 100000. Each value is the maximum of 1000 trials for each value of q

conclusions from the results. We attempted to alleviate this issue by performing a large number of trials and using large values of N wherever practical. However, we were limited by high computational costs and run times. This was especially the case when running experiments involving the L_1 minimization algorithm, as its optimization problem is much more computationally expensive than the DRA. Because of this, only 100 trials were used in the threshold finding algorithm rather than 1000 as stated in the above. The values of N used for L_1 minimization were also much smaller than were used for the DRA. These factors make the results for L_1 minimization even more inconsistent, so more care must be taken when drawing conclusions from the following data.

We used Corollary 2.3 to find an estimate for the Bourgain constant in the following method: Let $f : \mathbb{Z}_N \to [0,1]$ be randomly generated with a uniform distribution. Given some $q > 2$, let $S \subset \mathbb{Z}_N$ be a randomly chosen subset with size $\lceil N^{\frac{2}{q}} \rceil$ and let 1_S be its indicator function. Let $h = \widehat{f1_S}$. With our normalization of the Fourier transform, this is equivalent to constructing a signal such that its Fourier transform is supported in S. We then compute $\frac{\|f\|_{L^q(\mu)}}{\|f\|_{L^2(\mu)}}$, which we will call the Bourgain ratio. We repeat this computation many times with different randomly generated f and S, and take the maximum of these values to be our estimate of $C(q)$. By Corollary 2.3,

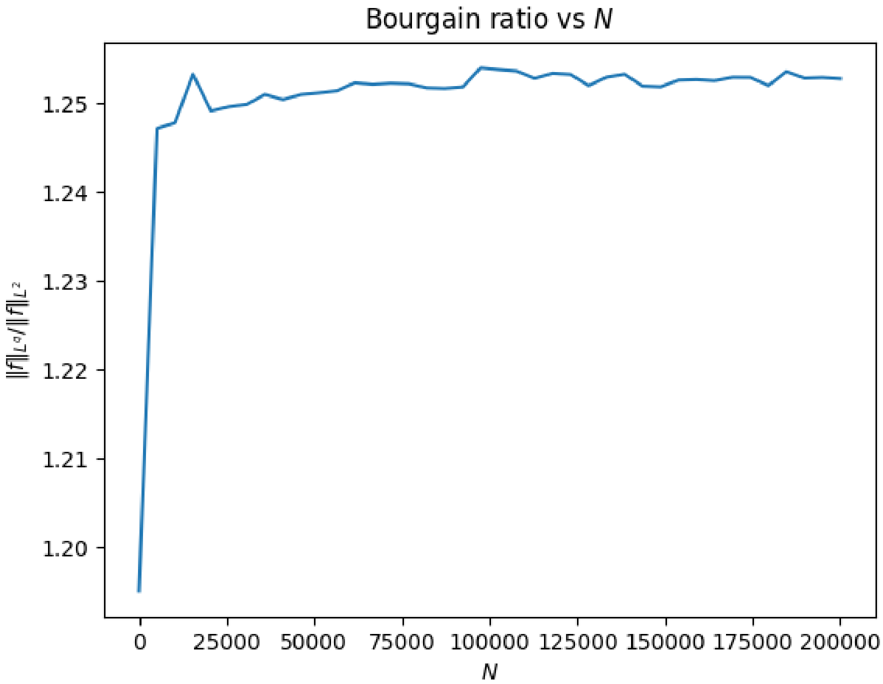

Figure 2: Bourgain ratio with respect to N. N varies from 20 to 200000. q is fixed at 3.

this ratio is always bounded above by the Bourgain constant. Because of this, our estimate cannot actually be equal to the Bourgain constant. However, by computing this many times, we hope to get sufficiently close to the true value of $C(q)$. The result of this estimate are shown in Figure 1.

As a sanity check, the same experiment was done at varying values of N rather than q. Since the Bourgain constant should be independent of N, the ratio should be constant. Indeed, as shown in Figure 2, the resulting value is close to 1.25 for all N with a q of 3. There are some fluctuations, especially for small N, but the value is still relatively stable.

We implemented the Direct Rounding Algorithm as described in Section 2 for the recovery of a single signal. Then, for a given set of conditions N, q, and $|E|$, we perform the recovery for a randomly generated f matching the conditions 1000 times. We consider a set of conditions satisfactory for recovery if all 1000 attempts successfully recovered their respective signal. Recovering 1000 out of 1000 attempts does not truly guarantee that recovery will always be successful, but it is close enough for the purposes of our analysis. As we can see from Figure 3, there is a specific cutoff point where for all $|E|$ less than that point, all signals were successfully recovered.

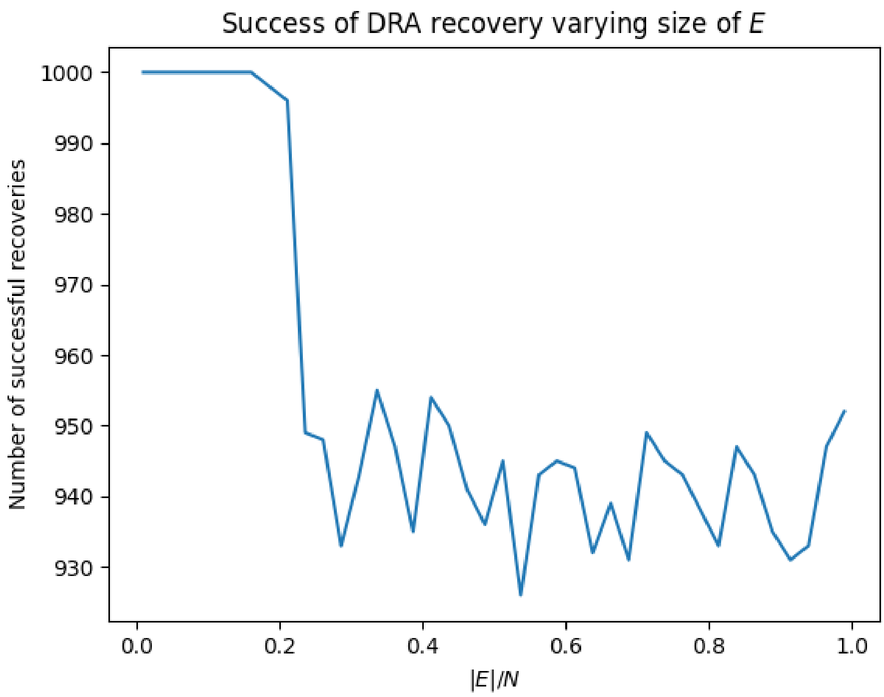

Figure 3: Number of successful recoveries for a given size of E. N is fixed at 5000. q is fixed at 3.

Let $c = \frac{|E|}{N}$. For a given N and q, we wish to find c such that for all $|E| < cN$, recovery is guaranteed. To do this, we used a binary search algorithm to find the largest $|E|$ with which all of the trials we run are able to achieve successful recovery. Let $e_1 = \lceil \frac{N}{2} \rceil$ be our starting point. For iteration n, let $\delta_n = \lfloor \frac{N}{2^{n+1}} \rfloor$ We then run the recovery on 1000 random signals for $|E| = e_n$. If all 1000 attempts were successful, then $e_{n+1} = e_n + \delta_n$. If any attempts were unsuccessful, then $e_{n+1} = e_n - \delta_n$. We continue this loop until $\delta_n < 1$ and let $\frac{e_n}{N}$ be our estimate for c. At this point, further iterations would not be productive, as $|E|$ must be an integer.

This algorithm was performed using the DRA to determine a threshold of recovery for the DRA in terms of N and q. This was compared to the conditions found by Donoho and Stark in Theorem 2.1 and that found by Iosevich and Mayeli in Theorem 2.4, which guarantee that signal recovery is possible. Taking $|S|$ as a generic set with size $\lceil N^{\frac{2}{q}} \rceil$, equation (2.1) can be rewritten in terms of c as

$$c \leq \frac{1}{2N^{\frac{2}{q}}}$$

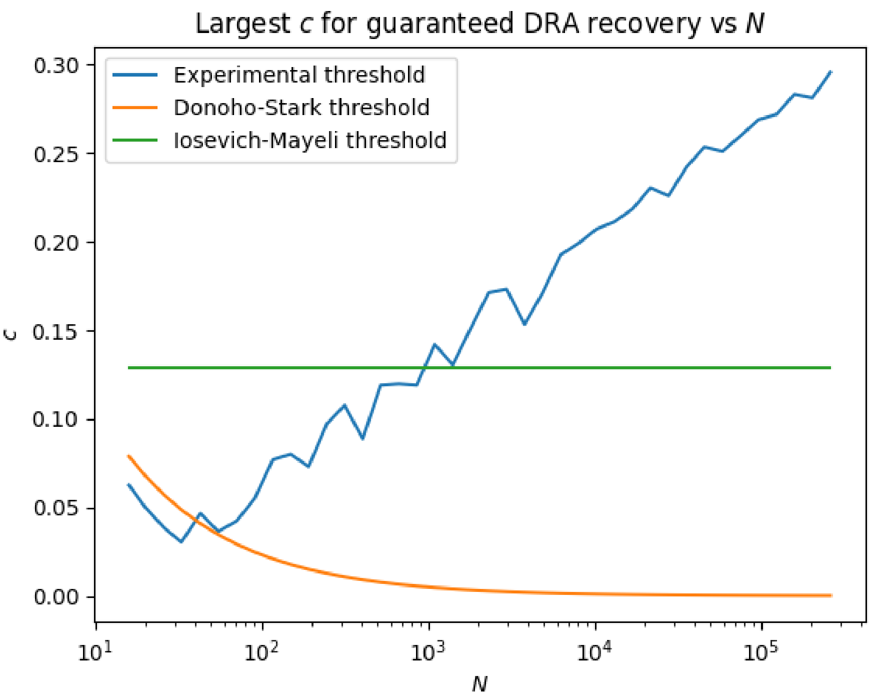

Figure 4: Threshold for successful recovery with the DRA with with respect to N. q is fixed at 3. Donoho-Stark threshold is based on (2.1). Iosevich-Mayeli threshold is based on (2.3).

Similarly, equation (2.3) can be rewritten as

$$c \leq \frac{1}{2(C(q))^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}}$$

In place of $C(q)$ we will use the estimate of the Bourgain constant obtained in the first experiment described in this section.

In Figure 4, we see that the experimentally derived threshold for recovery for the DRA is roughly proportional to $\log N$. For the most part, it is bounded below by the Donoho-Stark threshold, which confirms Theorem 2.5. The experimental threshold does not seem to correlate with the Iosevich-Mayeli threshold, with the Iosevich-Mayeli threshold being greater for $N < 1000$ and the experimental threshold being greater for $N > 1000$. This is reasonable, as Theorem 2.4 does not refer to any specific recovery method, only that signal recovery is possible under its conditions. It is plausible that signals can be recovered at the Iosevich-Mayeli threshold for small N, just not with the DRA.

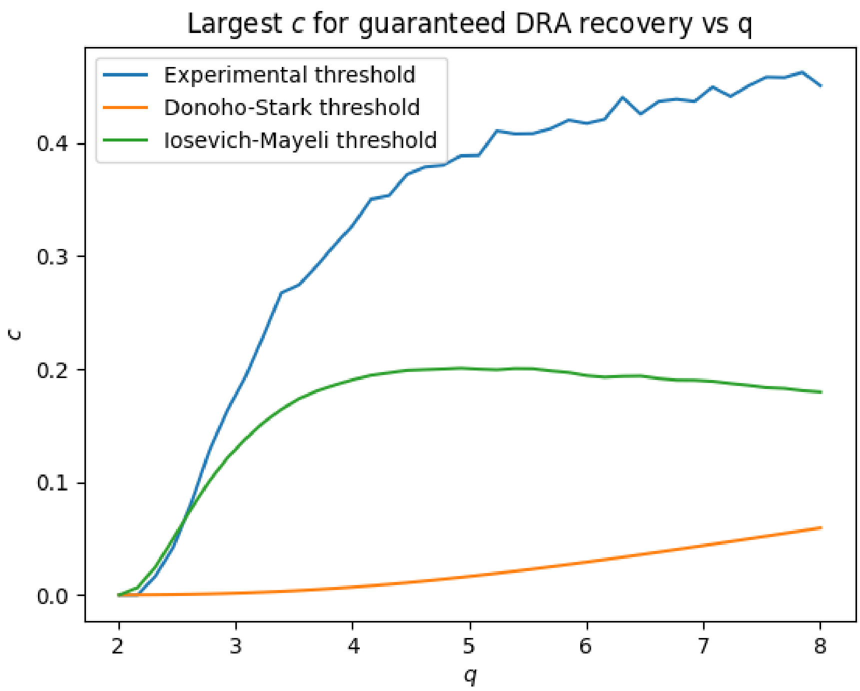

Figure 5: Threshold for successful recovery with the DRA with respect to q. N is fixed at 5000. Donoho-Stark threshold is based on (2.1). Iosevich-Mayeli threshold is based on (2.3).

In Figure 5, we can see that the experimental threshold is also bounded below by the Donoho-Stark threshold with respect to q. The experimental threshold was also mostly greater than the Iosevich-Mayeli threshold, except for $q < 2.62$.

The same procedure was repeated using the L_1 minimization technique, using the same binary search algorithm to find a suitable c and comparing the results to Theorem 2.4 and Theorem 2.6. In the same way as the above thresholds, (2.4) can be rewritten as

$$c \leq \frac{1}{4(C(q))^6}$$

The performance of the L_1 minimization algorithm was also tested with both random signals and random binary signals. In Figure 7, the experimentally derived threshold is not meaningfully changed by restricting to binary signals. Thus, we can also compare the efficacy of L_1 minimization with that of the DRA.

Figure 6 shows the experimentally derived threshold of recovery for the L_1 minimization algorithm. As expected by Theorem 2.6, it is bounded below by the Logan threshold. The L_1 minimization threshold is also much greater than the DRA threshold derived above, showing that it is a more effective method of signal

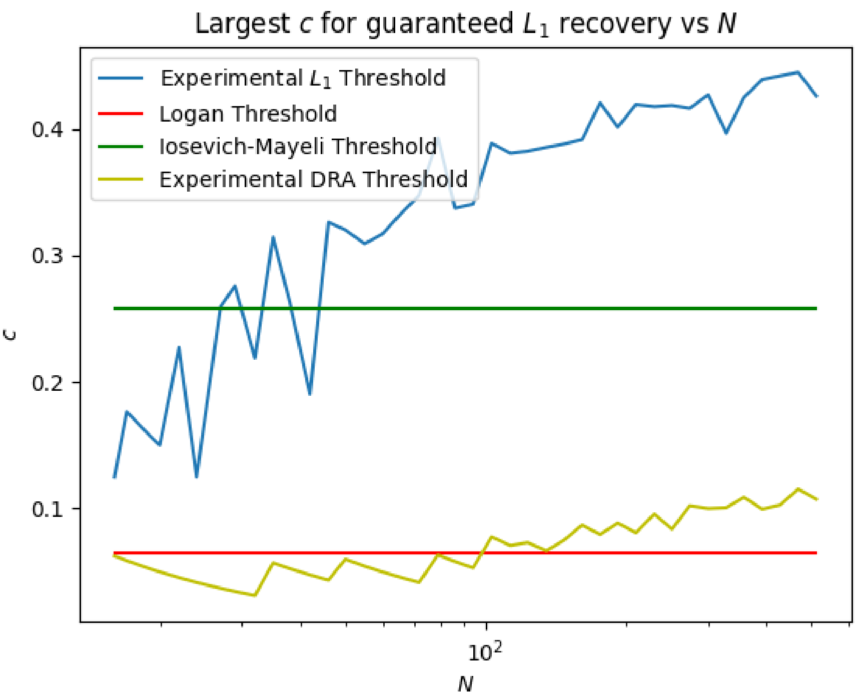

Figure 6: Threshold for successful recovery with L_1 minimization with respect to N. q is fixed at 3. Logan threshold is based on (2.4). Iosevich-Mayeli threshold is based on (2.3). Also compared to the threshold for the DRA for the same values of N and q.

recovery overall even when only processing binary signals. While the Iosevich-Mayeli Threshold is still greater than the experimental threshold for small N, with L_1 minimization the cutoff is at around $N = 32$, rather than $N = 1000$ for the DRA. This shows that L_1 minimization is a very effective signal recovery method, though there are still signals that are recoverable by other means that it is unable to recover.

In Figure 8, we see that the experimental L_1 threshold is almost fully bounded below by the Logan threshold, the Iosevich-Mayeli threshold, and the experimental DRA threshold. However, there is an exception for q very close to 2. If $q = 2$, then a signal can only be reliably recovered if $|E| = 0$ and thus the signal itself is 0. At the same time, the Bourgain constant must be 1 as $q = 2$ implies that $\|f\|_{L^q(\mu)} = \|f\|_{L^2(\mu)}$. However, by Theorem 2.6, the L_1 minimization algorithm is able to recover signals if $c \leq \frac{1}{4(C(q))^6} = \frac{1}{4}$.

All of the above algorithms were implemented and tested in Python. As we are performing experiments on random signals, our data will have some variance and different trials will have varying values. As such, it is difficult to draw exact conclusions from the results. We attempted to alleviate this issue by performing

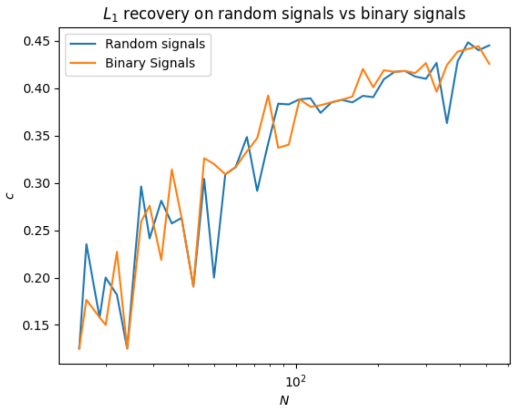

Figure 7: Threshold for successful recovery with L_1 minimization with respect to N applied to random signals and binary signals. q is fixed at 3.

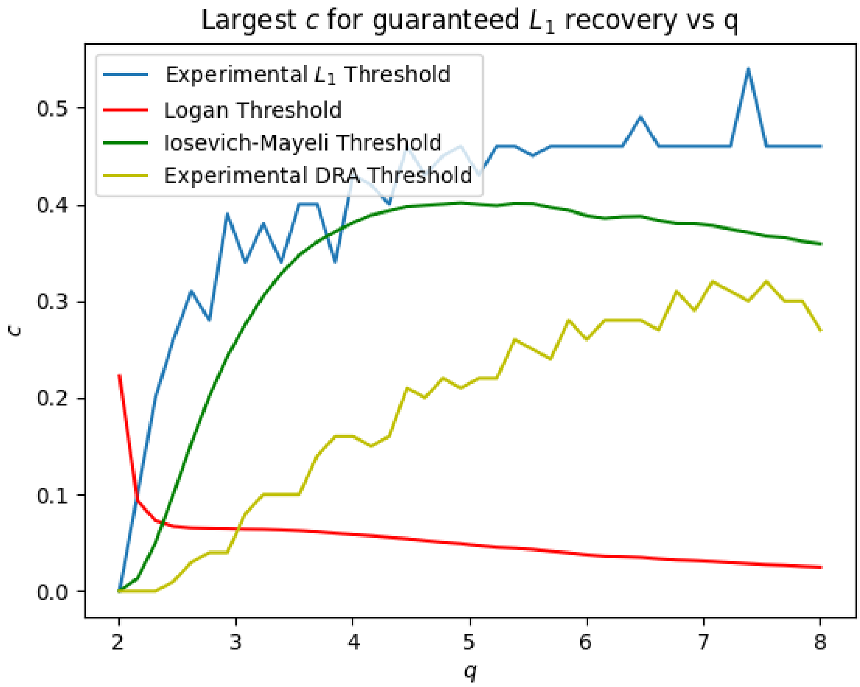

Figure 8: Threshold for successful recovery with L_1 minimization with respect to q. N is fixed at 100. Logan threshold is based on (2.4). Iosevich-Mayeli threshold is based on (2.3). Also compared to the threshold for the DRA for the same values of N and q

a large number of trials and using large values of N wherever practical. However, we were limited by high computational costs and run times. This was especially the case when running experiments involving the L_1 minimization algorithm, as its optimization problem is much more computationally expensive than the DRA. Because of this, only 100 trials were used in the threshold finding algorithm rather than 1000 as stated in the above. The values of N used for L_1 minimization were also much smaller than were used for the DRA. These factors make the results for L_1 minimization even more inconsistent, so more care must be taken when drawing conclusions from the following data.

References

[1] J. Bourgain. Bounded orthogonal systems and the Λ(p)-set problem. *Acta Math*, 1989.

[2] D. Donoho and P. Stark. Uncertainty principle and signal processing. *SIAM Journal of Applied Math*, 1989.

[3] A Iosevich and A. Mayeli. Uncertainty principles, restriction, bourgain's Λq theorem, and signal recovery. *Applied and Computational Harmonic Analysis 76*, 2025.

[4] B. F. Logan. Properties of high-pass signals. *Ph.D. thesis, Department of Electrical Engineering, Columbia University, New York*, 1965.