Investigation of Type D ASEP Corresponding to $\mathcal{U}_q(\mathfrak{so}_8)$

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1 Abstract

The particle system labeled the asymmetric simple exclusion process (ASEP) involves two classes of particles. A generator of ASEP can be produced from both a probabilistic construction and an algebraic construction [KLLPZ20], which result in the same generator for Type A Lie algebras [Kua19]. However, specifying to Type D ASEP, the methods do not produce the same generator when considering the Type D $\mathcal{U}_q(\mathfrak{so}_6)$ [BEPS24]; in fact, the algebraic method constructs a generator allowing for more states than are possible in the typical Type D ASEP. We test whether the same conclusion holds for $\mathcal{U}_q(\mathfrak{so}_8)$ by decomposing and analyzing various $\mathcal{U}_q(\mathfrak{so}_8)$-modules in order to construct a fused Type D ASEP generator from a central element of $\mathcal{U}_q(\mathfrak{so}_8)$. We conclude that neither the original statement nor this more flexible statement can hold for such a generator.

2 Introduction

Type D ASEP [KLLPZ20] is an interacting particle system that generalizes Spitzer's ASEP [Spi70]. ASEP consists of two classes of particles jumping on a lattice with one restriction: while two particles of different classes can exist at a site, two particles of the same class cannot. The specification of Type D refers to the system's relationship with the Type D Lie Algebra $\mathfrak{so}_{2n}$.

One main finding of [BEPS24] was that the algebraically-produced generator of Type D ASEP allowed for all states at a site except for four particles of the same class, while the probabilistic generator only allowed for two or less particles of the same class at a site. The generator was constructed from a central element of $\mathcal{U}_q(\mathfrak{so}_6)$. We question whether a generator constructed from a central element of $\mathcal{U}_q(\mathfrak{so}_8)$ preserves this more flexible structure.

For background, [BEPS24] deduces that the probablistic and algebraic approaches of constructing a generator do not agree for Type D ASEP, even though the two approaches produce the same generator for Type A ASEP as studied in [CGRS14; CRV14; CGRS16; Kua16; Kua17; Kua18; Kua21]. The probablistic method utilizes stochastic fusion, as defined in as defined in [Kua19]. On the other hand, the algebraic construction consisted of applying a ground state transformation to a Casimir element in the second tensor power of an irreducible representation of $\mathcal{U}_q(\mathfrak{so}_6)$.

This paper proceeds as follows: background information and notation are discussed in Sections 2.1 and 2.2, main results are listed in Section 3, proofs are detailed in Section 4, and figures are included in Section 6.

2.1 The Quantum Group $\mathcal{U}_q(\mathfrak{so}_8)$

A Lie group is a set G which is both a group and a manifold, with the two structures agreeing as multiplication is differentiable and inversion is smooth. We focus on SO_8.

Definition 1. The special orthogonal Lie group SO_8 is the multiplicative group with elements

$$SO_8 = \left\{X \in M_{8\times 8}(\mathbb{C}) : XX^T = I, \det X = 1\right\}$$

Since SO_8 is a subset of $GL_n(\mathbb{C})$, we define its corresponding Lie algebra, $\mathfrak{so}_8$, as a vector space such that there exists a neighborhood $U \subset SO_8$ of 1_{SO_8} and a neighborhood $\mathfrak{u} \subset \mathfrak{so}_8$ of $0_{\mathfrak{so}_8}$ such that

$$\log : U \cap SO_8 \to \mathfrak{u} \cap \mathfrak{so}_8$$

and

$$\exp : \mathfrak{u} \cap \mathfrak{so}_8 \to U \cap SO_8$$

are inverses of each other. Given $x \in \mathfrak{so}_8$, we then require that $\exp(x)\exp(x)^T = 1$. Rearranging, we must have that $x + x^T = 0$, and thus $\mathfrak{so}_8 = \{x \in M_8(\mathbb{C}) | x + x^T = 0\}$. We define $\mathfrak{so}_8$ using the following equivalent formulation.

Definition 2. The special orthogonal Lie algebra $\mathfrak{so}_8$ is the Lie algebra with elements

$$\mathfrak{so}_8 = \left\{ \begin{bmatrix} A & C \\ -C^T & B \end{bmatrix} : A, B, C \in M_{4\times 4}(\mathbb{C}), A = -A^T, B = -B^T \right\}$$

However, the usual multiplication in $\mathfrak{so}_8$ is not well defined, resulting in the construction of $\mathcal{U}(\mathfrak{so}_8)$, which is an algebra generated by elements in $\mathfrak{so}_8$ which abides by certain relations such as a commutator relation.

Definition 3. The Universal Enveloping Algebra of $\mathfrak{so}_8$, denoted $\mathcal{U}(\mathfrak{so}_8)$, is generated by $\{E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, H_1, H_2, H_3, H_4\}$ and the following relations:

$$[E_i, F_i] = H_i, \qquad 1 \le i \le 4$$

and

$$E_l^2 E_j + E_j E_l^2 = 2E_l E_j E_l; \quad F_l^2 F_j + F_j F_l^2 = 2F_l F_j F_l$$

for (l, j) such that $a_{lj} = -1$ in the following Cartan matrix; all other elements commute.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

A central element of $\mathcal{U}(\mathfrak{so}_8)$ can be manipulated to become a generator of a symmetric particle system. In order to use the procedure of this symmetric case, we keep the algebraic structure of $\mathcal{U}(\mathfrak{so}_8)$ by instead working in $\mathcal{U}_q(\mathfrak{so}_8)$ [Dri85];[Jim85]. Allowing this parameter $0 < q \le 1$ allows the particles to drift, creating an asymmetric particle system; $q = 1$ is the equivalent of no drift.

Definition 4. The q-deformed quantum group $\mathcal{U}_q(\mathfrak{so}_8)$ is generated by $\{E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4, q^{H_1}, q^{H_2}, q^{H_3}, q^{H_4}\}$ and the relations:

$$\begin{aligned} [E_i, F_i] &= \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \\ q^{H_i} E_j &= q^{\alpha_i \cdot \alpha_j} E_j q^{H_i} \\ q^{H_i} F_j &= q^{-\alpha_i \cdot \alpha_j} F_j q^{H_i} \end{aligned}$$

for $1 \leq i, j \leq 4$ and

$$E_l^2 E_k + E_k E_l^2 = (q + q^{-1}) E_l E_k E_l; \quad F_l^2 F_k + F_k F_l^2 = (q + q^{-1}) F_l F_k F_l$$

for all (l, k) such that $a_{lk} = -1$ in the following Cartan matrix; all other elements commute.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

For a matrix H in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_8$, we also define $L_i \in \mathfrak{h}^*$ as a map from H to $H_{i,i}$.

Along with this structure, we also desire that $\mathcal{U}_q(\mathfrak{so}_8)$ be a Hopf algebra. Therefore, we equip the quantum group with a coproduct, a counit, and an antipode as follows.

Definition 5. We define the coproduct Δ, counit ϵ, and antipode S of $\mathcal{U}_q(\mathfrak{so}_8)$ to be

$$\begin{aligned} &\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i, && \epsilon(E_i) = 0, && S(E_i) = -E_i q^{H_i} \\ &\Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{-H_i}, && \epsilon(F_i) = 0, && S(F_i) = -q^{-H_i} F_i \\ &\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}, && \epsilon(q^{H_i}) = 1, && S(q^{H_i}) = q^{-H_i}. \end{aligned}$$

for $1 \leq i \leq 4$.

Next, in order to compute the representation of an element in the second symmetric tensor power module of $\mathcal{U}_q(\mathfrak{so}_8)$, we must first define its fundamental representation. Note that all of the relations listed in Definition 4 will hold in this representation, but so do some extra relations that are not true in general.

Definition 6. A fundamental representation of $\mathcal{U}_q(\mathfrak{so}_8)$ is defined to be the subset of $M_{8\times 8}(\mathbb{R}[q, q^{-1}])$ generated from the elements of Definition 4, represented in the following chart. $E_{i,j}$ indicates the 8×8 matrix with a 1 in the $(i, j)^{th}$ entry and zeroes elsewhere, and $\text{diag}(x_1, ...x_8)$ represents the 8×8 diagonal matrix with the elements $(x_1, ..., x_8)$ along the diagonal. We then define q^{-H_i} to be the multiplicative inverse of q^{H_i}. Finally, we denote this fundamental representation by V.

	E_i	F_i	q^{H_i}
$i=1$	$E_{1,2}-E_{6,5}$	$E_{2,1}-E_{6,5}$	$\text{diag}(q,q^{-1},1,1,q^{-1},q,1,1)$
$i=2$	$E_{2,3}-E_{7,6}$	$E_{3,2}-E_{6,7}$	$\text{diag}(1,q,q^{-1},1,1,q^{-1},q,1)$
$i=3$	$E_{3,4}-E_{8,7}$	$E_{4,3}-E_{7,8}$	$\text{diag}(1,1,q,q^{-1},1,1,q^{-1},q)$
$i=4$	$E_{3,8}-E_{4,7}$	$E_{7,4}-E_{8,3}$	$\text{diag}(1,1,q,q,1,1,q^{-1},q^{-1})$

Finally, we briefly review the definition of a highest weight module; recall that a highest weight module is necessarily an irreducible representation.

Definition 7. We define P to be the weight lattice of $\mathcal{U}_q(\mathfrak{so}_8)$, with $\lambda \in P$ called a weight. Additionally, v_λ is called a highest weight vector if

$$E_1 v_\lambda = E_2 v_\lambda = E_3 v_\lambda = E_4 v_\lambda = 0.$$

Given a highest weight vector v_λ, then, we say that a $\mathcal{U}_q(\mathfrak{so}_8)$-module M is a highest weight module if $M = \{Av_\lambda | A \in \mathcal{U}_q(\mathfrak{so}_6)\}$.

2.2 Crystal Bases

The basis vectors of the fundamental representation as defined in Definition 6 were the standard basis vectors of $\mathbb{R}^8$, $\{e_1, ..., e_8\}$. However, the basis vectors of other $\mathcal{U}_q(\mathfrak{so}_8)$-modules, particularly those worked with in Section 4, will be q-deformed. In general, decomposing a tensor product of two representations into a direct sum of irreducible representations is difficult, so we would like to simplify this problem. By letting $q \to 0$, though, the q-deformed basis vectors match a sum of tensor products of $\{e_1, ..., e_8\}$. This describes the motivation and concept of crystal bases; we now must define them rigorously so that manipulating these crystal basis vectors corresponds to manipulating those of the target $\mathcal{U}_q(\mathfrak{so}_8)$-module.

Definition 8. A q-integer is an element of $\mathbb{C}(q)$ of the form

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

If m is an integer, we then let $[m]_q! = [m]_q[m-1]_q...[1]_q$. Using this notation, we are then able to define the action

$$F_i^{(m)} u = \frac{1}{[m]_q!} F_i^m u$$

where F_i^m denotes applying F_i m times.

As proven in [HK02], any weight vector u can be written as follows.

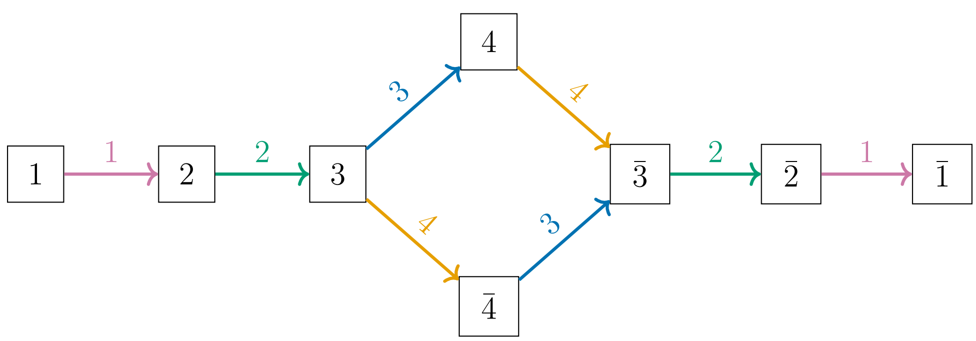
$$u = \sum_{m=0}^{n} F_i^{(m)} u_m$$

We now would like the ability to permute tensor products of basis vectors in a natural way.

Definition 9. For each $i \in \{1, 2, 3, 4\}$, we define the Kashiwara operators $\tilde{E}_i$ and $\tilde{F}_i$ by

$$\tilde{E}_i = \sum_{k=1}^{N} F_i^{(k-1)} u_k \quad \text{ and } \quad \tilde{F}_i = \sum_{k=0}^{N} F_i^{(k+1)} u_k.$$

For example, direct computation yields the following graph which shows the action of the Kashiwara operators $\{\tilde{F}_i\}$ on the fundamental representation V. Denote $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_4 = e_4$, $v_{\bar{4}} = e_8$, $v_{\bar{3}} = -e_7$, $v_{\bar{2}} = e_6$, $v_{\bar{1}} = -e_5$. An arrow superscripted with i from v_j to v_k symbolizes that $\tilde{F}_i v_j = v_k$. Finally, $\boxed{j}$ represents v_j, and $\boxed{\bar{j}}$ represents $v_{\bar{j}}$.



The action of $\{\tilde{E}_i\}$ would be represented by flipping every arrow.

We are now ready to define the crystal lattice, crystal base, and crystal limit.

Definition 10. Let W be an irreducible highest weight $\mathcal{U}_q(\mathfrak{so}_8)$-module with highest weight λ and highest weight vector v_λ. We then define the crystal lattice of W, denoted $\mathcal{L}(\lambda)$, to be a free submodule of W spanned by $\{\tilde{F}_{i_1}...\tilde{F}_{i_n} v_\lambda\}$ with each $i_k \in \{1, 2, 3, 4\}$ and $n \geq 0$.

Definition 11. Let W, v_λ be as in Definition 10. Then, we define the crystal base $\mathcal{B}(W)$ of W to be

$$\{\tilde{F}_{i_1}...\tilde{F}_{i_n}e_1 \otimes e_1 + q\mathcal{L}(\lambda)|i_k \in \{1, 2, 3\}, n \geq 0\} \setminus \{0\}.$$

Note that the crystal lattice $\mathcal{L}(W)$ is a submodule over the localization of $\mathbb{C}[q]$ at (q), and is a principal ideal domain. Thus, modding by its unique maximal ideal $q\mathcal{L}$, we obtain the equivalence of letting $q \to 0$.

Definition 12. Formally, taking the crystal limit of $v \in \mathcal{L}(W)$ is the action of projecting v to $\bar{v} \in \mathcal{L}/q\mathcal{L}$.

Finally, we are able to discuss crystal bases, as defined in [HK02].

Definition 13. A pair $(\mathcal{L}, \mathcal{B})$ is a crystal base of a $\mathcal{U}_q(\mathfrak{so}_8)$-module M if it satisfies all of the following conditions:

1. $\mathcal{L}$ is a crystal lattice of M
2. $\mathcal{B}$ is a $\mathbb{C}$-basis of $\mathcal{L}/q\mathcal{L}$
3. $\mathcal{B} = \bigsqcup \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap \mathcal{L}/q\mathcal{L}$, $\lambda \in P$
4. $\tilde{E}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{F}_i\mathcal{B} \subset \mathcal{B} \cup \{0\}$ for each $i = 1, 2, 3$
5. $\tilde{F}_i b_1 = b_2$ if and only if $b_1 = \tilde{E}_i b_2$ for every $b_1, b_2 \in \mathcal{B}$, $i = 1, 2, 3$.

3 Results

The symmetries of quantum groups has been used to construct a form of ASEP multiple times [CGRS14; BS15b; BS15a; CGRS16; BS16; Kua16; Kua17; Kua18; KLLPZ20; RLY23; BEPS24]. As such, the main focus of this paper is to compare this algebraic construction of a Markov generator to the typical ASEP. The resulting generators have in different cases been shown to match [Kua19] and differ [BEPS24] from ASEP, with the differing generator allowing for even more state spaces than those in ASEP. The main result is as follows.

Theorem 1. *It is impossible to algebraically produce a Markov generator from [KLLPZ20]'s central element in the representation $W \otimes W$ that will be equal to a probabilistic generator of Type D ASEP up to a constant. Specifically, any algebraically-produced Markov generator can only have a maximum block size of* 12*, which would be impossible to achieve in a non-trivial probabilistic generator of Type D ASEP.*

In order to compute this generator, we first need to construct and decompose $W \otimes W$.

Proposition 1. *Let W be the* 35*-dimensional subspace of V such that $W = V(2L_1)$. Basis vectors for this subspace can be found in Section 6.1. Then, W is an irreducible $\mathcal{U}_q(\mathfrak{so}_8)$ module.*

Proposition 2. *$W \otimes W$ decomposes as follows into a direct sum of irreducible, highest weight representations, and is* 1225*-dimensional.*

$$W \otimes W \cong V(4L_1) \oplus V(3L_1 + L_2) \oplus V(2L_1 + 2L_2) \oplus V(2L_1) \oplus V(L_1 + L_2) \oplus V(0).$$

Proposition 3. *$\pi_{W \otimes W}(C)$ can be blocked into one 41×41 block, twenty-four 18×18 blocks, eight 12×12 blocks, sixteen 6×6 blocks, ninety-six 4×4 blocks, twenty-four 3×3 blocks, forty-eight 2×2 blocks, and eight 1×1 blocks.*

In particular, note that the two biggest block sizes of $\pi_{W \otimes W}(C)$ are 41×41 and 18×18. Since the blocks of a Markov generator represent a single communicating class, it is sufficient to only look at these classes of blocks; after performing a ground state transformation, if these blocks have dimension larger or smaller than the size of the biggest communicating class of the probabilistic Type D ASEP, then the generator constructed from $\pi_{W \otimes W}(C)$ cannot be equal up to a constant to the communicating classes of the nontrivial probabilistic analog.

Proposition 4. *The largest block of $\pi_{W \otimes W}(C)$ after a ground state transformation can be at most 12×12.*

Proposition 5. *The largest communicating class allowed by a probabilistically-constructed generator of Type D ASEP contains 16 states.*

With these last two propositions combined, we will be able to prove Theorem 1.

4 Proofs

4.1 Constructing W

We begin by proving Proposition 1. Recall the tensor product theorem, as stated on pages 77-78 of [HK02]:

Theorem 2. *[HK02]. Let $V(\lambda)$, $V(\mu)$ be $\mathcal{U}_q(\mathfrak{so}_8)$-modules with corresponding crystal bases $(\mathcal{L}(\gamma), \mathcal{B}(\gamma))$, $(\mathcal{L}(\mu), \mathcal{B}(\mu))$. Then, $(\mathcal{L}(\gamma) \otimes \mathcal{L}(\mu), \mathcal{B}(\gamma) \times \mathcal{B}(\mu))$ is a crystal basis of $V(\lambda) \otimes V(\mu)$, where the action of $\tilde{F}_i$ is defined by:*

$$\tilde{F}_i(b_1 \otimes b_2) = \begin{cases} \tilde{F}_i b_1 \otimes b_2 & \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{F}_i b_2 & \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

Using this theorem, we are able to draw the crystal graphs of $V \otimes V$ as shown in Appendix 6.2. Figure 2 shows W on this crystal graph, and Figure 3 shows the full decomposition of $V \otimes V$. Since taking the tensor product of two vectors of weights L_i and L_j, respectively, results in a vector of weight $L_i + L_j$, we are able to see from Figure 3 that

$$V \otimes V \cong V(2L_1) \oplus V(L_1 + L_2) \oplus V(0).$$

We are now able to prove Proposition 1.

Proof. Since $\mathcal{U}_q(\mathfrak{so}_8)$ admits a Cartan decomposition, $V(2L_1)$ can be computed by applying compositions of the generators F_is to the highest weight vector $e_1 \otimes e_1$. Direct computation shows that $e_1 \otimes e_1$ is annihilated by every generator E_i, and thus $e_1 \otimes e_1$ is indeed a highest weight vector.

Therefore, every basis vector of W must be a composition of the generators F_is with $e_1 \otimes e_1$. Since the crystal graph in Figure 2 demonstrates the possible actions of each F_i on each vector $e_i \otimes e_j$, we may proceed by tracing the arrows from $e_1 \otimes e_1$ to each desired point on the graph. For example, letting $\pi_V(F_{ij})$ denote $\pi_V(F_iF_j)$, we have that

$$\begin{aligned}
\pi_V F_1(e_1 \otimes e_1) &= (1 \otimes F_1)(e_1 \otimes e_1) + (F_1 \otimes q^{H_1})(e_1 \otimes e_1) \\
&= e_1 \otimes e_2 + e_2 \otimes (q^{-1} e_1) \\
&= e_1 \otimes e_2 + q^{-1} e_2 \otimes e_1 \\
\pi_V(F_{12})(e_1 \otimes e_1) =& \pi_V(F_2)(e_1 \otimes e_2 + q^{-1} e_2 \otimes e_1) \\
&= (e_1 \otimes e_3 + 0 \otimes e_2) + q^{-1}(e_2 \otimes 0 + e_3 \otimes e_1) \\
&= e_1 \otimes e_3 + q^{-1} e_3 \otimes e_1 \\
\pi_V(F_{123421124321}(e_1 \otimes e_1) &= (q^2 + 2 + q^{-2})(q^2 + 2 + q^{-2}) e_5 \otimes e_5
\end{aligned}$$

As seen in Appendix 6.1, this results in 35 distinct basis vectors. From [FH04], we are also equipped with a formula to calculate the dimension of any highest weight $\mathcal{U}_q(\mathfrak{so}_8)$-module, namely

$$\dim(V(\lambda)) = \frac{1}{4320} \sum_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 8 - j - i)$$

where $\lambda = \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 + \lambda_4 L_4$. Plugging in to find $\dim(V(2L_1))$, we see that

$$\begin{aligned}
\dim(V(2L_1)) &= \frac{1}{4320} \sum_{1 \leq i < j \leq 4} (\lambda_i - \lambda_j + j - i)(\lambda_i + \lambda_j + 8 - j - i) \\
&= \frac{12}{4320} [(\lambda_1 + 1)(\lambda_1 + 5)][(\lambda_1 + 2)(\lambda_1 + 4)][(\lambda_1 + 3)(\lambda_1 + 3)] \\
&= \frac{(21)(24)(25)(12)}{4320} = 35.
\end{aligned}$$

Finally, note that a highest weight module is always irreducible, and thus the 35-dimensional $V(2L_1)$, which is thus spanned by the linearly independent bases vectors of W, must be an irreducible $\mathcal{U}_q(\mathfrak{so}_8)$-module which is equal to W. □

As an example, after some very tedious computation of sparse 64×64 matrices, one can represent [KLLPZ20]'s central element C of $\mathcal{U}_q(\mathfrak{so}_8)$ in W, which results in

$$(q^{10} + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-10})\mathrm{Id}_{35}.$$

[KLLPZ20]'s central element C can be found in Appendix 6.3.

4.2 Decomposing $W \otimes W$

If we were to decompose $W \otimes W$ using 2, we would need to visualize the direct product $\mathcal{B}(2L_1) \times \mathcal{B}(2L_1)$ and then draw all disjoint cycles. Due to the complexity of $\mathcal{B}(2L_1)$ as shown in Figure 3, we introduce a method using Young tableaux instead. Following the convention of [HK02], $\mathcal{B}(2L_1)$ can be described as

$$\mathcal{B}(2L_1) = \mathcal{B}\left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array} \right) := \mathcal{B}(\mathcal{Y})$$

with the two blocks in the first row representing the weight $2L_1$. We now have the notation needed to approach Lemma 1.

Lemma 1. *$W \otimes W$ admits the following decomposition into irreducible representations:*

$$W \otimes W \cong V(4L_1) \oplus V(3L_1 + L_2) \oplus V(2L_1 + 2L_2) \oplus V(2L_1) \oplus V(L_1 + L_2) \oplus V(0).$$

The sum of the dimensions of these irreducible representations is $1225 = \dim(W \otimes W)$.

Proof. Note that the vectors included in W as can be read off of Figure 3 exactly satisfy the following:

$$\mathcal{B}(2L_1) = \left\{ \begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \;\middle|\; b \succeq a \right\}.$$

Using the English notation and convention, by the tensor product rule for Young diagrams [HK02],

$$\mathcal{B}(\mathcal{Y}) \otimes \mathcal{B}(\mathcal{Y}) \cong \bigoplus_{b_1 \otimes b_2 \in W} \mathcal{B}(\mathcal{Y}[b_1, b_2]).$$

We must compute the right hand side. Removing every degenerate Young diagram, we see that the only possible options for $\mathcal{B}(\mathcal{Y}[b_1, b_2]$ with $b_1 \otimes b_2 \in W$ are:

$$\begin{aligned}
\mathcal{B}(\mathcal{Y}[v_1, v_1]) &= \mathcal{B}\left(\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array} \right) \\
\mathcal{B}(\mathcal{Y}[v_2, v_1]) &= \mathcal{B}\left(\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \right) \\
\mathcal{B}(\mathcal{Y}[v_2, v_2]) &= \mathcal{B}\left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array} \right) \\
\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_1]) &= \mathcal{B}\left(\begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array} \right) \\
\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_2]) &= \mathcal{B}\left(\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} \right) \\
\mathcal{B}(\mathcal{Y}[v_{\bar{1}}, v_{\bar{1}}]) &= \mathcal{B}(\varnothing).
\end{aligned}$$

Therefore, computing the right hand side, we must have that

$$\mathcal{B}(\mathcal{Y})\otimes\mathcal{B}(\mathcal{Y}) \cong \mathcal{B}\left(\ydiagram{4}\right)\oplus\mathcal{B}\left(\ydiagram{3,1}\right)\oplus\mathcal{B}\left(\ydiagram{2,2}\right)\oplus\mathcal{B}\left(\ydiagram{2}\right)\oplus\mathcal{B}\left(\ydiagram{1,1}\right)\oplus\mathcal{B}(\varnothing).$$

Reading a Young diagram having j blocks in the first row and i blocks in the second row as being $V(jL_1 + iL_2)$, we can interpret this decomposition as

$$W \otimes W \cong V(4L_1) \oplus V(3L_1 + L_2) \oplus V(2L_1 + 2L_2) \oplus V(2L_1) \oplus V(L_1 + L_2) \oplus V(0).$$

Finally, to compute the dimension of each irreducible $\mathcal{U}_q(\mathfrak{so}_8)$-module, we again use [FH04]'s formula as in the proof of Proposition 1 and determine that

$$\begin{aligned}\dim(W \otimes W) &= \sum_\lambda \dim V(\lambda)\\ &= 294 + 567 + 300 + 35 + 28 + 1\\ &= 1225.\end{aligned}$$

The order of the summed dimensions is the same as the order that the irreducible representations are listed in in the above decomposition.

□

We proceed to investigating the weight spaces of $W \otimes W$, as this will help block $W \otimes W$. We approach this proof of Proposition 3 combinatorially.

Proof. Note that, since taking the tensor product of two elements corresponds to adding their weights, the only possible weights that an element of $W \otimes W$ could have are

$$\{0, 2L_i, L_i + L_j, 2L_i + L_j + L_k, 2L_i + 2L_j, 3L_i + L_j, 4L_i, L_i + L_j + L_k\}$$

for distinct $i, j, k \in \{1, 2, 3, 4\}$ as elements of W can only have weights of forms $2L_i, L_i + L_j$, or 0. We proceed by first finding the number of weight spaces of each form, which dictates the number of corresponding blocks, then by counting the dimension of a weight space of each form, which corresponds to the size of each block. To demonstrate the process, we compute the dimension and number of weight spaces corresponding to $L_i + L_j$, which is objectively the most involved case.

First, the number of weight spaces corresponding to $L_i + L_j$ can be computed by determining the number of possible vector weights in $W \otimes W$ that could be of that form. Note that we are equivalently counting pairs (L_i, L_j) such that L_i, L_j are signed and i, j are distinct.

Therefore, we have $4 \cdot 3$ possible pairs of distinct i, j, and 4 ways to sign (L_i, L_j). Since order does not matter in this case (ie, $L_i + L_j = L_j + L_i$), we must have $4 \cdot 4 \cdot 3/2$, or 24, weight spaces of form $L_i + L_j$.

Next, to determine the dimension of each such weight space, we fix (i, j) and count the possible ways to tensor two elements in W that would result in $L_i + L_j$. For this weight, we must consider three cases, since for $k \neq i, j$,

$$L_i + L_j = (2L_i) + (L_i - L_j) = (2L_j) + (L_j - L_i) = (L_i + L_j) + 0 = (L_i + L_k) + (L_j - L_k).$$

Since (i, j) is already determined, the first two expressions can only be formed two ways each, as order matters. In other words, since we are taking the tensor product of W with itself, $(2L_i) + (L_i + L_j)$ and $(L_i + L_j) + 2L_i$ are the result of two different tensor products of elements, and thus are distinct in this counting scheme. Moving on to $(L_i + L_j) + 0$, note that there are three ways to achieve 0 in Figure 2: $v_{\bar{1}} \otimes v_1$, $v_{\bar{2}} \otimes v_2$, and $v_{\bar{3}} \otimes v_3$. Therefore, there are $3 \cdot 2$ ordered pairs of elements that tensor to be of this weight. Finally, we count tuples of (i, j, k) all distinct, of which there are two, note that swapping induces $2 \cdot 2$, and finally note that $(L_i + L_k) + (L_j - L_k)$ is not equivalent to $(L_i - L_k) + (L_j + L_k)$, thus adding one more factor of two. Summing all of these combinations, we see that there are $2 + 2 + 6 + 8 = 18$ possible ways to tensor two elements of W that would result in a weight of $L_i + L_j$, and thus each weight space will have dimension 18.

Thus, there are 24 weight spaces of dimension 18, and thus twenty-four 18×18 blocks in $\pi_{W \otimes W}(C)$. Similarly, there is one weight space of dimension 41 ($\lambda = 0$), eight weight spaces of dimension 12 ($\lambda = 2L_i$), sixteen weight spaces of dimension 6 ($\lambda = L_i + L_j + L_k + L_\ell$), ninety-six weight spaces of dimension 4 ($\lambda = 2L_i + L_j + L_k$), twenty-four weight spaces of dimension 3 ($\lambda = 2L_i + 2L_j$), forty-eight weight spaces of dimension 2 ($\lambda = 3L_i + L_j$), and eight weight spaces of dimension 1 ($\lambda = 4L_i$). These numbers correspond to the number of blocks and block sizes in the statement of Proposition 3. □

4.3 Significance of the blocks of $\pi_{W \otimes W}(C)$

As discussed in [BEPS24], a ground state transformation maps a Hamiltonian element H to a matrix whose rows sum to 0, and in our case is of the form $G^{-1}HG - a\mathrm{Id}$ where $a = \pi_{W \otimes W}(C)_{1,1}$ and G is a diagonal matrix. In order to show that $\pi_{W \otimes W}(C)$ results in a generator of a Markov process that could not possibly be Type D ASEP, it is sufficient to prove that this generator will not have a block that is the size of the biggest block in a probabilistically-constructed generator, as each block represents a communicating class.

Thus, we only consider performing a ground state transformation on the 41×41 and 18×18 blocks of $\pi_{W \otimes W}(C)$ which greatly reduces computation.

The ground state transformation that was uniquely determined for Type A Lie algebras is no longer uniquely determined for those of Type D due to the more complicated structure, as was proven in [BEPS24]. Therefore, we must prove that there cannot exist any possible ground state transformation that would map $\pi_{W \otimes W}(C)$ to a Type D ASEP generator. This discussion supports Proposition 4.

Proof. Per the above commentary, $\pi_{W \otimes W}(C)$ has one block of size 41 and twenty-four of size 18. Recall that, in order to have a valid Markov generator, we can only have negative values on the diagonal; if there is a negative value in the $(i,j)^{th}$ position, we remove either the i^{th} row and column, the j^{th} row and column, or both. We will work directly with the signs of the elements in the 41×41 and 18×18 blocks in order to argue that, no matter what diagonal matrix $\pi_{W \otimes W}(C)$ is conjugated by, there will still be negative entries in many rows and columns. We would then have to remove enough rows and columns that the size of each block after the ground state transformation would be less than 16×16.

Consider a diagonal matrix G conjugating a matrix H. Then,

$$(G^{-1}HG)_{i,j} = G_{i,i}^{-1} H_{i,j} G_{j,j}.$$

In particular, if $H_{i,j}$ and $H_{j,i}$ have opposite signs, then there is no choice of diagonal matrix G that would cause both $(G^{-1}HG)_{i,j}$ and $(G^{-1}HG)_{j,i}$ to agree in sign, resulting in the removal of at least one row and column.

In Python, we compute $\pi_{W \otimes W}(C)$ using Sympy, then compare the signs of the $(i,j)^{th}$ and $(j,i)^{th}$ values in the 41×41 and each of the 18×18 blocks. After adjusting for errors due to multiplication of small values, we see that at least 27 rows and columns have to be deleted from the 41×41 block and at least 5 have to be deleted from each of the 18×18 blocks. The code ran to complete this computation can be found on GitHub at `https://github.com/lgstolberg/SeniorThesis`, and the curious reader is invited to reach out to lgstolberg@gmail.com with any questions about this program.

Therefore, any ground state transformation of the 41×41 and 18×18 blocks can only result in a block of maximum size 14×14. □

Finally, we take a closer look at the Type D ASEP that corresponds to $\mathcal{U}_q(\mathfrak{so}_8)$. $\mathcal{U}_q(\mathfrak{so}_8)$ corresponds to allowing up to three particles of two different classes at each stochastically fused site, ie the possible states at a stochastically fused site are as follows.

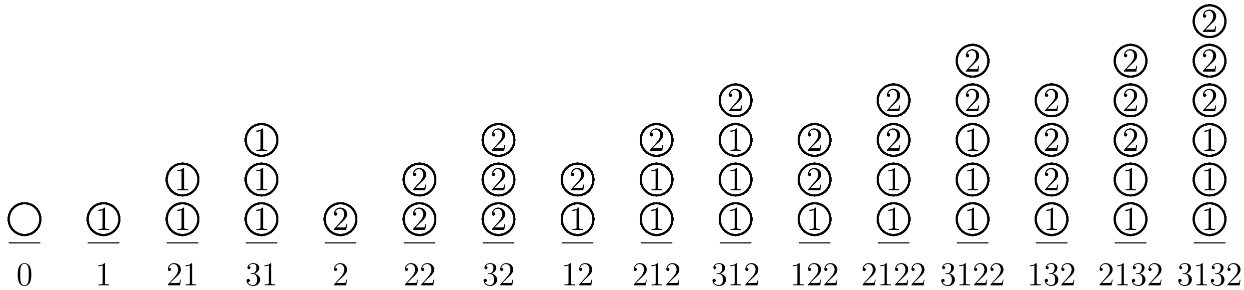


Figure 1: Possible states at one stochastically fused site

More information on stochastic fusion and this correspondence can be found in [Kua19];[BEPS24].

Then, since each communicating class is represented by a block of a Markov generator, we want to find the largest communicating class that can exist on two lattice sites. Drawing all possible communicating classes shows that the biggest communicating class involves all six particles at once, ie

$$\Big(\langle 0,3132\rangle,\langle 2,3122\rangle,\langle 22,312\rangle,\langle 31,32\rangle,\langle 132,21\rangle,\langle 2132,1\rangle,\langle 12,2122\rangle,\langle 212,122\rangle,$$
$$\langle 3132,0\rangle,\langle 3122,2\rangle,\langle 312,22\rangle,\langle 32,31\rangle,\langle 21,132\rangle,\langle 1,2132\rangle,\langle 2122,12\rangle,\langle 122,212\rangle\Big).$$

However, note that this communicating class has 16 states, and thus the largest block of a probabilistically-produced Markov generator is 16×16. Intuitively, this is a result of the fact that, if there are more particles involved in a communicating class, then there are more possible combinations of said particles at each site, and thus more possible states in the communicating class. This proves Proposition 5.

Finally, we have shown that the biggest block possible in a probabilistically-produced Markov generator for the Type D ASEP corresponding to $\mathcal{U}_q(\mathfrak{so}_8)$ is 16×16, but algebraically producing a Markov generator from a central element of $\mathcal{U}_q(\mathfrak{so}_8)$ can never result in a block that is strictly greater than 14×14. Therefore, the algebraically-produced generator fails to match a Type D ASEP generator in a new way: it is degenerate in comparison. This proves Theorem 1.

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6 Appendix

6.1 Basis vectors of W

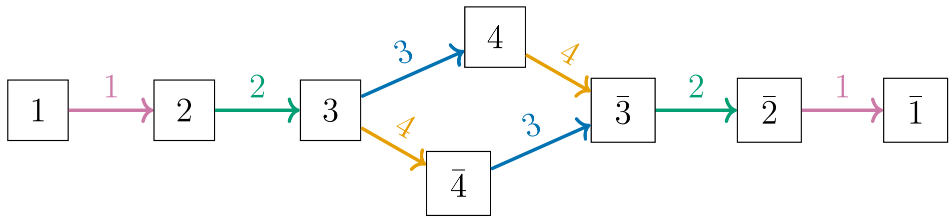
The basis vectors of W are as follows.

$$
\begin{aligned}
\{&e_1 \otimes e_1,\ e_1 \otimes e_2 + q^{-1} e_2 \otimes e_1,\ e_1 \otimes e_3 + q^{-1} e_3 \otimes e_1,\ e_1 \otimes e_4 + q^{-1} e_4 \otimes e_1,\ -e_1 \otimes e_8 - q^{-1} e_8 \otimes e_1,\\
&e_1 \otimes e_7 + q^{-1} e_7 \otimes e_1,\ -e_1 \otimes e_6 - q^{-1} e_6 \otimes e_1,\ e_1 \otimes e_4 - q^{-1} e_2 \otimes e_6 + q^{-2} e_4 \otimes e_1 - q^{-1} e_6 \otimes e_2,\\
&(q+q^{-1}) e_2 \otimes e_5 + (1+q^{-2}) e_5 \otimes e_2,\ (q+q^{-1}) e_3 \otimes e_5 + (1+q^{-2}) e_5 \otimes e_3,\\
&(q+q^{-1}) e_4 \otimes e_5 + (1+q^{-2}) e_5 \otimes e_4,\ -(q+q^{-1}) e_8 \otimes e_5 - (1+q^{-2}) e_5 \otimes e_8,\\
&(q+q^{-1}) e_7 \otimes e_5 + (1+q^{-2}) e_5 \otimes e_7,\ -(q+q^{-1}) e_6 \otimes e_5 - (1+q^{-2}) e_5 \otimes e_6,\\
&(q^2+2+q^{-2}) e_5 \otimes e_5,\ (q+q^{-1}) e_2 \otimes e_2,\ (q+q^{-1}) e_2 \otimes e_3 + (1+q^{-2}) e_3 \otimes e_2,\\
&(q+q^{-1}) e_2 \otimes e_4 + (1+q^{-2}) e_4 \otimes e_2,\ -(q+q^{-1}) e_2 \otimes e_8 - (1+q^{-2}) e_8 \otimes e_2,\\
&(q+q^{-1}) e_2 \otimes e_7 + (1+q^{-2}) e_7 \otimes e_2,\\
&-(q+q^{-1}) e_2 \otimes e_6 - (q^{-1}+q^{-3}) e_6 \otimes e_2 + (1+q^2)(e_3 \otimes e_7 + e_7 \otimes e_3),\\
&-(q^2+2+q^{-2}) e_3 \otimes e_6 - (q+2q^{-1}+q^{-3}) e_6 \otimes e_3,\\
&-(q^2+2+q^{-2}) e_4 \otimes e_6 - (q+2q^{-1}+q^{-3}) e_6 \otimes e_4,\\
&(q^2+2+q^{-2}) e_8 \otimes e_6 + (q+2q^{-1}+q^{-3}) e_6 \otimes e_8,\\
&(q^2+2+q^{-2}) e_7 \otimes e_6 + (q+2q^{-1}+q^{-3}) e_6 \otimes e_7,\\
&(q^3+3q++3q^{-1}+q^{-3}) e_6 \otimes e_6,\ (q^2+2+q^{-2}) e_3 \otimes e_3\ (q^3+3q+3q^{-1}+q^{-3}) e_4 \otimes e_4,\\
&(q^2+2+q^{-2}) e_3 \otimes e_4 + (q+2q^{-1}+q^{-3}) e_4 \otimes e_3,\ (q^4+4q^2+6+4q^{-2}+q^{-4}) e_7 \otimes e_7,\\
&-(q^2+2+q^{-2}) e_3 \otimes e_8 - (q+2q^{-1}+q^{-3}) e_6 \otimes e_3,\ (q^3+3q+3q^{-1}+q^{-3}) e_8 \otimes e_8,\\
&(q^2+2+q^{-2}) e_3 \otimes e_7 + (1+2q^{-2}+q^{-4}) e_7 \otimes e_3 - (q+2q^{-1}+q^{-3})(e_4 \otimes e_8 + e_8 \otimes e_4),\\
&-(q^2+3+3q^{-2}+q^{-4}) e_7 \otimes e_8 - (q^3+3q+3q^{-1}+q^{-3}) e_8 \otimes e_7,\\
&(q^2+3+3q^{-2}+q^{-4}) e_7 \otimes e_3 + (q^3+3q+3q^{-1}+q^{-3}) e_3 \otimes e_7\}
\end{aligned}
$$

Note that this is a 35-dimensional invariant subspace of $\mathrm{Sym}_q^2(\mathbb{R}^8)$.

6.2 Crystal Graph of $V \otimes V$

Below are the crystal graphs of $V \otimes V$. The first highlights just W.



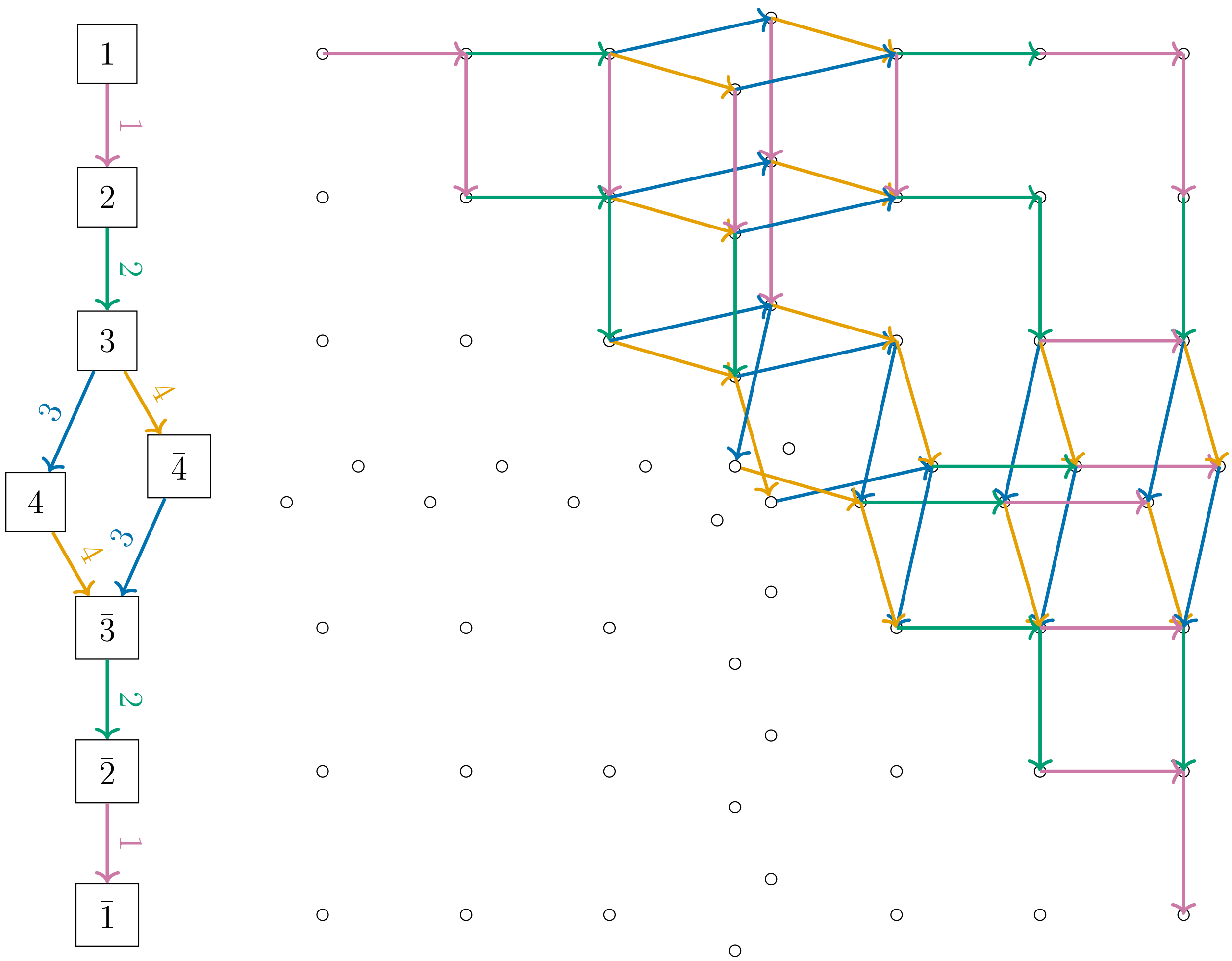


Figure 2: Crystal graph of $V(2L_1)$ in the representation $V \otimes V$

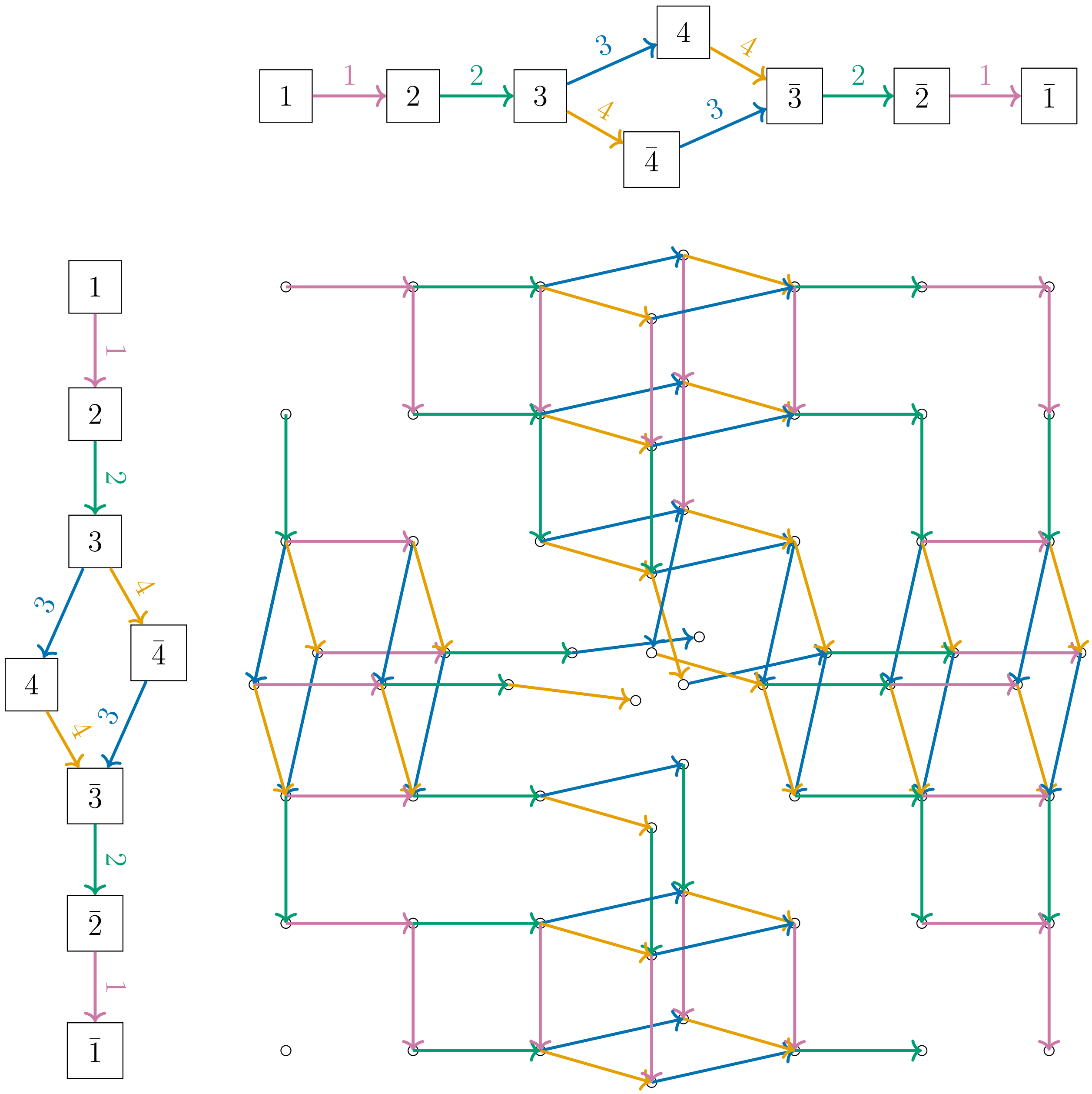


Figure 3: Crystal graph of decomposition of the representation $V \otimes V$

6.3 [KLLPZ20]'s Central Element C

Define $r = q + q^{-1}$. We will use the shorthand $F_{12} = F_1 F_2$. Then, we define C as follows

$$\begin{aligned}
C = {} & q^{-6-2H_1-2H_2-H_3-H_4} + q^{-4-2H_2-H_3-H_4} + q^{-2-H_3-H_4} + q^{H_3-H_4} \\
& + q^{H_4-H_3} + q^{2+H_3+H_4} + q^{4+2H_2+H_3+H_4} + q^{6+2H_1+2H_2+H_3+H_4} \\
& + \frac{r^2}{q^5} F_1 q^{-H_1-2H_2-H_3-H_4} E_1 + \frac{r^2}{q^5}(qF_{12} - F_{21}) q^{-H_1-H_2-H_3-H_4}(qE_{21} - E_{12}) \\
& + \frac{r^2}{q^5}(q^2F_{123} - qF_{132} - qF_{213} + F_{321}) q^{-H_1-H_2-H_4}(q^2E_{321} - qE_{213} - qE_{132} + E_{123}) \\
& - \frac{r^2}{q^5}(q^2F_{124} - qF_{142} - qF_{241} + F_{421}) q^{-H_1-H_2-H_3}(q^2E_{421} - qE_{241} - qE_{142} + E_{124}) \\
& - \frac{r^2}{q^5} A_1 q^{-H_1-H_2} A_4 - \frac{r^2}{q^5} A_5 q^{-H_1} A_8 - \frac{r^4}{q^4} A_9 A_{10} \\
& + \frac{r^2}{q^3} F_2 q^{-H_2-H_3-H_4} E_2 + \frac{r^2}{q^3}(qF_{23} - F_{32}) q^{-H_2-H_4}(qE_{32} - E_{23}) \\
& - \frac{r^2}{q^3}(qF_{24} - F_{42}) q^{-H_2-H_3}(qE_{42} - E_{24}) \\
& - \frac{r^2}{q^3}(q^2F_{234} - qF_{324} - qF_{423} + F_{432}) q^{-H_2}(q^2E_{432} - qE_{324} - qE_{423} + E_{234}) \\
& - \frac{r^4}{q^2}((q^2+1)F_{2342} - qF_{3242} - qF_{2423})((q^2+1)E_{2342} - qE_{3242} - qE_{2423}) \\
& - \frac{r^2}{q^3} A_7 q^{H_1} A_6 + \frac{r^2}{q} F_3 q^{-H_4} E_3 - \frac{r^2}{q} F_4 q^{-H_3} E_4 \\
& - r^4 F_3 F_4 E_4 E_3 - \frac{r^2}{q}(q^2F_{432} - qF_{324} - qF_{423} + F_{234}) q^{H_2}(q^2E_{234} - qE_{324} - qE_{423} + E_{432}) \\
& - \frac{r^2}{q} A_3 q^{H_1+H_2} A_2 - r^2 q F_4 q^{H_3} E_4 - r^2 q(qF_{42} - F_{24}) q^{H_2+H_3}(qE_{24} - E_{42}) \\
& - r^2 q(q^2F_{421} - qF_{241} - qF_{142} + F_{124}) q^{H_1+H_2+H_3}(q^2E_{124} - qE_{142} - qE_{241} + E_{421}) \\
& + r^2 q F_3 q^{H_4} E_3 + r^2 q(qF_{32} - F_{23}) q^{H_2+H_4}(qE_{23} - E_{32}) \\
& + r^2 q(q^2F_{321} - qF_{213} - qF_{132} + F_{123}) q^{H_1+H_2+H_4}(q^2E_{123} - qE_{132} - qE_{213} + E_{321}) \\
& + r^2 q^3 F_2 q^{H_2+H_3+H_4} E_2 + r^2 q^3(qF_{21} - F_{12}) q^{H_1+H_2+H_3+H_4}(qE_{12} - E_{21}) + r^2 q^5 F_1 q^{H_1+2H_2+H_3+H_4} E_1,
\end{aligned}$$

given that

$$A_1 = q^3F_{1234} - q^2F_{2314} - q^2F_{3124} - q^2F_{1423} + qF_{4213} + qF_{3241} + qF_{4132} - F_{4321},$$

$$A_2 = q^3 E_{1234} - q^2 E_{2314} - q^2 E_{3124} - q^2 E_{1423} + qE_{4213} + qE_{3241} + qE_{4132} - E_{4321}.$$
$$A_3 = q^3 F_{4321} - q^2 F_{4132} - q^2 F_{4213} - q^2 F_{3241} + qF_{1423} + qF_{2314} + qF_{3124} - F_{1234},$$
$$A_4 = q^3 E_{4321} - q^2 E_{4132} - q^2 E_{4213} - q^2 E_{3241} + qE_{1423} + qE_{2314} + qE_{3124} - E_{1234},$$
$$A_5 = q^4 F_{12342} + F_{23421} + q^2 F_{42132} + q^2 F_{24123} - qF_{23214} + q^2 F_{23124} - (q^3 + q)F_{23142},$$
$$A_6 = q^4 E_{12342} + E_{23421} + q^2 E_{42132} + q^2 E_{24123} - qE_{23214} + q^2 E_{23124} - (q^3 + q)E_{23142},$$
$$A_7 = F_{12342} + q^4 F_{23421} + q^2 F_{42132} + q^2 F_{24123} - q^3 F_{23214} + q^2 F_{23124} - (q^3 + q)F_{23142},$$
$$A_8 = E_{12342} + q^4 E_{23421} + q^2 E_{42132} + q^2 E_{24123} - q^3 E_{23214} + q^2 E_{23124} - (q^3 + q)E_{23142},$$
$$A_9 = (-q^3 - q)F_{121342} - \frac{q^4}{(q^2+1)^2} F_{223141} + q^2 F_{143122} - q^2 F_{122341} - \frac{q^2(q^4+q^2+1)}{(q^2+1)^2} F_{412231},$$
$$A_{10} = (-q^3 - q)E_{121342} - \frac{q^4}{(q^2+1)^2} E_{223141} + q^2 E_{143122} - q^2 E_{122341} - \frac{q^2(q^4+q^2+1)}{(q^2+1)^2} E_{412231}.$$

We note that a couple of typos from [KLLPZ20] were corrected in this restating of the element. As can be computed directly, this element acts as

$$(q^8 + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-8})\mathrm{Id}_8$$

when represented in $\mathbb{R}^8$, and as

$$(q^{10} + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-10})\mathrm{Id}_{35}$$

when represented in W.

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