Thesis

A Study of the Classical and Stochastic Heat Equations

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1 Abstract

The thesis systematically explores the mathematical connection between classic heat equation and stochastic heat equations, particularly highlighting the role of semigroup theory in linking classical analysis with stochastic processes. Starting from fundamental concepts such as Banach spaces and linear operators, we first illustrate how strongly continuous semigroups and their infinitesimal generators provide a robust framework for analyzing and solving deterministic heat equations. Subsequently, randomness is introduced through the microscopic Symmetric Simple Exclusion Process (SSEP), and the stochastic heat equation is rigorously derived via macroscopic limits and central limit theorem arguments. Special emphasis is placed on the martingale formulation and mild solution representation, clarifying the precise emergence and physical significance of the noise coefficient.

2 Introduction

The heat equation is one of the most fundamental partial differential equations (PDEs) in mathematical physics, modeling the diffusion of heat or particles in a medium. Its classical form,

$$\partial_t u(t,x) = \Delta u(t,x),$$

has been extensively studied using tools from functional analysis, Fourier methods, and PDE theory. The deterministic nature of this equation assumes an idealized world, where the evolution of heat is perfectly smooth and predictable.

However, many real-world systems exhibit randomness at microscopic scales fluctuations due to individual particle interactions, thermal agitation, or random external forces. To incorporate these effects, one naturally considers stochastic counterparts to the heat equation. A prototypical example is the *Stochastic Heat Equation (SHE)*:

$$\partial_t u(t,x) = \Delta u(t,x) + \dot{W}(t,x),$$

where $\dot{W}(t,x)$ denotes space-time white noise. This equation plays a central role in the theory of Stochastic Partial Differential Equations (SPDEs), and also appears as a linearization of more complex systems, such as the Kardar-Parisi-Zhang (KPZ) equation.

A key goal of this thesis is to *bridge the classical and stochastic theories* of the heat equation. This connection is made precise through two complementary perspectives:

1. **Semigroup theory**, a deterministic tool from functional analysis, provides a robust framework for solving evolution equations such as the heat equation. It explains how unbounded operators like the Laplacian can generate continuous dynamics via a strongly continuous semigroup.

2. **Microscopic particle systems**, in particular the *Symmetric Simple Exclusion Process (SSEP)*, offer a probabilistic route to understanding macroscopic diffusion. Through rescaling limits and martingale techniques, one derives the stochastic heat equation as a fluctuation limit of SSEP, revealing the origin and structure of the noise term.

The thesis is structured as follows. In Chapter 2, we lay out the necessary functional analytic background, including Banach spaces, linear operators, and the definition of one-parameter strongly continuous semigroups. Chapter 3 applies this framework to analyze the classical heat equation, introducing the heat semigroup, its infinitesimal generator, and the Hille-Yosida theorem. In Chapters 4 and 5, we shift focus to the microscopic setting, constructing the SSEP, identifying its Markov generator, and deriving the stochastic heat equation via macroscopic scaling limits. Finally, Chapter 6 presents the martingale and mild formulations of the limiting SPDE, offering both analytical and probabilistic interpretations.

This work aims to illuminate the deep interplay between deterministic and stochastic analysis, highlighting how tools from semigroup theory, probability, and PDEs come together to describe complex diffusion phenomena across scales.

3 Foundations of Functional Analysis

In order to study the heat equation from an abstract perspective and to understand how randomness can be incorporated via stochastic analysis, it is essential to first establish a solid foundation in functional analysis. This chapter introduces the key concepts needed throughout the thesis, including Banach spaces, linear operators, and semigroup theory. We begin with the notion of completeness in normed spaces, move on to discuss the nature of unbounded operators such as the Laplacian, and conclude with a structured introduction to semigroups, which provide a natural framework for describing time evolution in both deterministic and stochastic systems.

3.1 Banach Spaces and Completeness

The concept of a Banach space lies at the heart of modern analysis. Intuitively, a Banach space is a normed vector space in which limits of converging sequences always exist within the space. More precisely, a Banach space is a vector space X over $\mathbb{R}$ or $\mathbb{C}$, equipped with a norm $\|\cdot\|$, such that every Cauchy sequence in X converges to an element of X. This property is known as completeness.

Examples:

- $\ell^p(\mathbb{N})$: sequences whose p-norm is finite;
- $L^p(\Omega)$: functions integrable to the p-th power over a domain Ω;
- $C([0, 1])$: continuous functions on a closed interval with the supremum norm.

Completeness ensures that analytic processes such as differentiation, integration, and limit-taking remain well-defined and well-behaved.

3.2 Linear Operators: Bounded and Unbounded

A linear operator $T : X \to X$ is said to be bounded if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. In normed spaces, boundedness is equivalent to continuity. The class of bounded linear operators is relatively easy to work with: they can be extended, composed, and analyzed using standard techniques.

However, in many applications—especially those involving differential equations—one must work with operators that are naturally unbounded. For example, the differential operator $\frac{d}{dx}$, or the Laplacian Δ, are not bounded when defined on spaces such as $L^2(\mathbb{R}^n)$. To make sense of such operators, one restricts their domain to a dense subspace of "sufficiently regular" functions.

3.3 Closed and Densely Defined Operators

Given that unbounded operators cannot be defined on all of X, it is important to specify their domain carefully. Let A be a linear operator defined on a subspace $D(A) \subset X$. The operator is said to be **closed** if, whenever a sequence $x_n \in D(A)$ satisfies $x_n \to x \in X$ and $Ax_n \to y \in X$, then $x \in D(A)$ and $Ax = y$.

Moreover, we often require that $D(A)$ be **dense** in X, meaning that any element in X can be approximated arbitrarily well by elements in $D(A)$. This property is crucial for defining adjoints, spectral theory, and, most importantly, for ensuring that semigroups generated by such operators are meaningful throughout the space.

3.4 Definition of Semigroups and Their Generators

Algebraic Semigroup

An algebraic semigroup is a set S equipped with a binary operation $\cdot$ satisfying the associative property:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \forall a, b, c \in S.$$

No identity or inverse is required. This structure appears in many areas of mathematics, including number theory, group theory, and dynamical systems.

Strongly Continuous Semigroups of Operators

Let us now move to the analytical setting. Let X be a Banach space. A family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called a strongly continuous one-parameter semigroup (or C_0-semigroup) if it satisfies the following three properties:

Definition 3.1 (Strongly Continuous Semigroup)**.** *A family of operators $\{T(t)\}_{t\geq 0}$ on a Banach space X is called a strongly continuous semigroup if:*

1. *$T(0) = I$, the identity operator on X;*
2. *$T(t+s) = T(t)T(s) \quad \forall t, s \geq 0$;*
3. *For all $x \in X$, $\lim_{t\to 0^+} \|T(t)x - x\| = 0$.*

The semigroup property models the time evolution of a system, and strong continuity ensures well-behaved limits with respect to time.

Definition 3.2 (Infinitesimal Generator)**.** *Given a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, its infinitesimal generator A is defined as:*

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

for all $x \in X$ for which this limit exists. The domain of A is given by

$$D(A) := \left\{ x \in X \mid \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists in } X \right\}.$$

The generator A is generally an unbounded operator, and plays a crucial role in linking the abstract semigroup with differential equations. The question of which operators can arise as generators will be answered in the next chapter via the Hille–Yosida theorem.

4 Semigroup Approach to the Heat Equation

The heat equation is one of the most fundamental examples of a partial differential equation exhibiting dissipative behavior. In this chapter, we develop a systematic approach to the heat equation using semigroup theory. Starting from the classical fundamental solution—known as the *heat kernel*—we introduce the heat semigroup as an operator acting on functions, and show that its generator is the Laplacian. To support this conclusion, we present and prove the Hille–Yosida theorem, which characterizes when an operator can generate a strongly continuous semigroup. We then derive the mild formulation of the heat equation using semigroup methods and demonstrate the uniqueness of the solution. Finally, we connect the heat semigroup to probabilistic concepts such as the Chapman–Kolmogorov identity and Brownian motion.

4.1 Heat Semigroup on $\mathbb{R}^n$ and the Hille–Yosida Theorem

We begin with the classical heat equation in $\mathbb{R}^n$:

$$\partial_t u(t,x) = \Delta u(t,x), \quad u(0,x) = f(x),$$

where Δ is the Laplacian on $\mathbb{R}^n$. The fundamental solution to this equation is the **heat kernel**:

$$p(t,x,y) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right), \quad t>0,\ x,y \in \mathbb{R}^n.$$

Definition 4.1 (Heat Kernel Representation). *Let $f \in C_b(\mathbb{R}^n)$ (bounded continuous functions). Define*

$$u(t,x) := \int_{\mathbb{R}^n} p(t,x,y) f(y)\, dy.$$

Then $u(t,x)$ solves the heat equation with initial condition $u(0,x) = f(x)$ in a classical sense.

Theorem 4.1 (Existence and Uniqueness of the Fundamental Solution). *Let $f \in C_b(\mathbb{R}^n)$. Then the function*

$$u(t,x) = \int_{\mathbb{R}^n} p(t,x,y) f(y)\, dy$$

is the unique classical solution to the heat equation with initial condition f. Moreover, $u(t,x) \to f(x)$ uniformly as $t \to 0^+$.

Proof. - *Existence*: Compute $\partial_t u$ and Δu under the integral. Differentiate under the integral sign (justified by dominated convergence) and verify that $\partial_t u = \Delta u$. - *Initial condition*: Show $\lim_{t\to 0^+} u(t,x) = f(x)$, using that $p(t,x,y) \to \delta_x(y)$ as $t \to 0^+$. - *Uniqueness*: Suppose two solutions agree at $t = 0$, then their difference solves the homogeneous equation with zero initial data, which must vanish. □

Using the heat kernel, we now define a linear operator $T(t)$ acting on functions f via convolution:

$$(T(t)f)(x) := \int_{\mathbb{R}^n} p(t,x,y) f(y)\, dy.$$

Definition 4.2 (Heat Semigroup)**.** *The family* $\{T(t)\}_{t\geq 0}$ *defined by the heat kernel is called the* heat semigroup*.*

We will now demonstrate that $\{T(t)\}$ is a strongly continuous semigroup whose generator is the Laplacian Δ.

Theorem 4.2 (Generator of the Heat Semigroup)**.** *Let $T(t)$ be defined as above. Then the generator A of $\{T(t)\}$ is the Laplacian:*

$$Af := \Delta f, \quad \textit{for all } f \in D(A),$$

where $D(A) = \{f \in C_b^2(\mathbb{R}^n)\}$.

Proof. Using the definition of the generator,

$$Af = \lim_{t\to 0^+} \frac{T(t)f - f}{t},$$

and applying a second-order Taylor expansion under the integral yields $Af = \Delta f$. □

To rigorously justify this generator result and ensure the well-posedness of the semigroup, we now state and prove the Hille–Yosida theorem, which characterizes generators of strongly continuous contraction semigroups.

Theorem 4.3 (Hille–Yosida Theorem)**.** *Let A be a linear operator on a Banach space X with domain $D(A) \subset X$. Then A is the generator of a strongly continuous contraction semigroup $\{T(t)\}_{t\geq 0}$ on X if and only if:*

1. *A is closed and densely defined;*
2. *For all $\lambda > 0$, the resolvent operator $R(\lambda, A) := (\lambda I - A)^{-1}$ exists and satisfies*
$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Proof. We divide the proof into two directions.

Suppose A generates a strongly continuous contraction semigroup $\{T(t)\}_{t\geq 0}$. Define the resolvent by Laplace transform:

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

Then:

$$(\lambda I - A)R(\lambda, A)x = x, \quad \text{and} \quad \|R(\lambda, A)\| \leq \int_0^\infty e^{-\lambda t}\|T(t)\| \, dt \leq \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}.$$

This shows that the resolvent exists and satisfies the required bound.

Suppose A is closed, densely defined, and satisfies the resolvent bound. For $x \in X$, define an approximate semigroup by the *Yosida approximation*:

$$A_\lambda := \lambda A R(\lambda, A), \quad T_\lambda(t) := e^{tA_\lambda}.$$

Each A_λ is a bounded operator, and $\|T_\lambda(t)\| \leq 1$ (contraction). Moreover, as $\lambda \to \infty$, one can show:

- $A_\lambda x \to Ax$ for all $x \in D(A)$,
- $T_\lambda(t)x \to T(t)x$ defines a semigroup,
- The limit satisfies $\frac{d}{dt}T(t)x = AT(t)x$.

This construction yields a strongly continuous contraction semigroup with generator A. □

4.2 Strong Continuity and the Generator (the Laplacian)

The family $\{T(t)\}$ defined via the heat kernel satisfies strong continuity. For any $f \in C_b(\mathbb{R}^n)$,

$$\lim_{t\to 0^+} \|T(t)f - f\|_\infty = 0,$$

due to the fact that $p(t,x,y) \to \delta_x(y)$ as $t \to 0^+$. The semigroup smooths f by convolving it with a Gaussian whose width shrinks to zero.

To verify this more precisely, one uses uniform continuity of f and localization of the heat kernel mass. As $t \to 0$, the kernel concentrates near x, so $T(t)f(x) \approx f(x)$, uniformly on compacts or in L^p.

The generator A of the semigroup is given by

$$Af := \lim_{t\to 0^+} \frac{T(t)f - f}{t},$$

for those f for which the limit exists. When $f \in C_c^\infty(\mathbb{R}^n)$, one can differentiate under the integral and verify

$$Af = \Delta f.$$

Thus, the Laplacian is the generator of the heat semigroup.

4.3 Mild Solutions and Uniqueness

Given a semigroup $T(t)$ with generator A, the mild solution to the abstract Cauchy problem

$$\frac{d}{dt}u(t) = Au(t), \quad u(0) = f$$

is defined as

$$u(t) := T(t)f.$$

In the case of the heat equation, this corresponds to

$$u(t,x) = \int_{\mathbb{R}^n} p(t,x,y)f(y)\,dy,$$

which solves the heat equation classically when $f \in C_b^2(\mathbb{R}^n)$, and is a mild solution otherwise.

Theorem 4.4 (Uniqueness of Mild Solutions)**.** *Let A generate a C_0-semigroup $T(t)$ on a Banach space X. Then the mild solution $u(t) = T(t)f$ is unique for each initial datum $f \in X$.*

Proof. If $u_1(t)$ and $u_2(t)$ are both mild solutions with initial data f, then their difference $v(t) := u_1(t) - u_2(t)$ satisfies $v(t) = T(t)(f - f) = 0$. Hence $u_1 = u_2$. □

4.4 Chapman–Kolmogorov Identity, Semigroup Property, and Connection to Brownian Motion

Let $S \subset \mathbb{R}^d$ be a measurable state space, and consider a family of functions $P(t,x;s,E)$, which represents the probability that a particle starting at position $x \in S$ at time t will be in the measurable set $E \subset S$ at time $s \geq t$. For a Markov process, this family must satisfy the following:

$$P(t,x;s,S) = 1, \quad P(t,x;s,\emptyset) = 0,$$

and for all $t < \tau \le s$, the Chapman–Kolmogorov equation holds:

$$P(t, x; s, E) = \int_S P(\tau, y; s, E) P(t, x; \tau, dy).$$

We often assume the process is homogeneous in time, so the transition kernel depends only on $s - t$. Then we write:

$$P(t, x; s, E) = P(s - t, x, E),$$

and the equation becomes:

$$P(t + \tau, x, E) = \int_S P(\tau, y, E) P(t, x, dy).$$

Markov Process. Let $\mathcal{B}$ be the σ-algebra of subsets of S. A family $P(t, x, E)$ is a **Markov process** on $(S, \mathcal{B})$ if it satisfies:

1. $P(t, x, E) \ge 0$, $P(t, x, S) = 1$;
2. $E \mapsto P(t, x, E)$ is σ-additive;
3. $x \mapsto P(t, x, E)$ is $\mathcal{B}$-measurable;
4. Chapman–Kolmogorov identity holds.

Semigroup from Transition Kernel. Define:

$$(T_t f)(x) := \int_S P(t, x, y) f(y) dy.$$

Then:

$$T_{t+s} = T_t \circ T_s,$$
$$\sup_{x \in S} |T_t f(x)| \le \sup_{x \in S} |f(x)|,$$

i.e., T_t is a contraction.

Brownian Motion. A Markov process is **spatially homogeneous** if:

$$P(t, i(x), i(E)) = P(t, x, E)$$

for all translations i. It is a Brownian motion if:

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{|x-y|>Q} P(t, x, y) dy = 0.$$

Theorem. Let B be the space of bounded uniformly continuous functions on $\mathbb{R}^d$. Then:

$$(T_t f)(x) := \int_{\mathbb{R}^d} P(t, x, y) f(y) dy, \quad T_0 f = f,$$

defines a contracting semigroup on B, if P satisfies the above.

Example. The heat kernel

$$P(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

satisfies all these conditions and thus represents a Brownian semigroup.

5 Microscopic Particle Systems — SSEP

In this chapter, we introduce the microscopic model that serves as the foundation for our derivation of the stochastic heat equation (SHE). The symmetric simple exclusion process (SSEP) is a paradigmatic interacting particle system whose macroscopic and fluctuation behaviors have been thoroughly studied. Through scaling and analysis of the density fluctuation field, we formally extract the SPDE structure underlying its large-scale limit.

5.1 Model Definition on $\mathbb{Z}$

The Symmetric Simple Exclusion Process (SSEP) is a continuous-time Markov process that describes a system of particles moving randomly on the integer lattice $\mathbb{Z}$, with the key rule that no two particles may occupy the same site — this is known as the *exclusion rule*.

More precisely, it takes value from the space defined by these coordinates(so call configuration space)of the system is

$$\Omega := \{0,1\}^{\mathbb{Z}},$$

where each site $x \in \mathbb{Z}$ can be either occupied ($\eta(x) = 1$) or empty ($\eta(x) = 0$). A configuration $\eta = (\eta(x))_{x\in\mathbb{Z}}$ specifies which sites are occupied at a given time.

The dynamics proceed as follows:

- Each particle waits an exponential time (with rate 1), then attempts to jump either to the left or right neighbor with equal probability.
- If the target site is already occupied, the jump is suppressed and the particle remains in place.
- If the target site is empty, the jump is performed.

This defines a system of random walks with interaction: the particles behave like independent symmetric random walkers, except that they cannot overlap.

We typically consider initial configurations drawn from a *Bernoulli product measure* ν_ρ, where each site is independently occupied with probability $\rho \in (0,1)$. That is, under ν_ρ, each $\eta(x)$ is an independent Bernoulli(ρ) random variable.

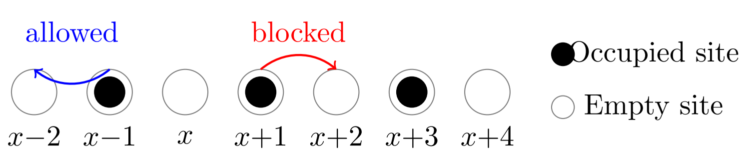

Figure 1: SSEP dynamics: particles attempt to jump left/right. A jump is only allowed if the target site is empty.

Feller Property (Definition). Let $C_0(\Omega)$ denote the space of continuous functions vanishing at infinity. A Markov process $(\eta_t)_{t\geq 0}$ on Ω is said to be a **Feller process** if its semigroup $T_t f(\eta) := \mathbb{E}_\eta[f(\eta_t)]$ maps $C_0(\Omega)$ into itself and is strongly continuous:

$$\lim_{t\to 0^+} \|T_t f - f\|_\infty = 0.$$

5.2 Exclusion Rule and Jump Dynamics

Each particle independently attempts to jump to the left or right with exponential waiting times of rate 1. The jump is carried out only if the destination site is empty, in accordance with the **exclusion rule**.

We denote by $\eta^{x,x+1}$ the configuration obtained by exchanging the occupation variables at sites x and $x+1$:

$$\eta^{x,x+1}(y) = \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

Lemma 5.1. *The SSEP defines a continuous-time Markov process on Ω with well-defined transition rates.*

Reference. See Liggett, *Interacting Particle Systems*, or Kipnis and Landim, *Scaling Limits of Interacting Particle Systems.* □

5.3 Markov Generator Formulation

The infinitesimal generator $\mathcal{L}$ of the SSEP acts on local (i.e., cylinder) functions $f : \Omega \to \mathbb{R}$ as:

$$(\mathcal{L}f)(\eta) = \sum_{x\in\mathbb{Z}} \left[f(\eta^{x,x+1}) - f(\eta) \right].$$

The process (η_t) with generator $\mathcal{L}$ defines a Feller process on Ω.

Sketch. The generator is local and bounded on local functions, satisfying the maximum principle. Its domain is dense in $C_0(\Omega)$, ensuring the Feller property. □

For each local function f, the process

$$M_t^f := f(\eta_t) - f(\eta_0) - \int_0^t \mathcal{L}f(\eta_s)\, ds$$

is a martingale. This is a direct consequence of Dynkin's formula.

5.4 From the Microscopic Dynamics to the Stochastic Heat Equation

Let $\rho \in (0,1)$ be the particle density. Define the fluctuation field acting on $\phi \in \mathcal{S}(\mathbb{R})$ by

$$Y_t^\epsilon(\phi) := \sqrt{\epsilon} \sum_{x\in\mathbb{Z}} \phi(\epsilon x)\, (\eta_{\epsilon^{-2}t}(x) - \rho).$$

Under the diffusive scaling $x \mapsto \epsilon^{-1}x$, $t \mapsto \epsilon^{-2}t$, the generator acts formally on Y_t^ϵ to yield

$$\partial_t Y = \Delta Y + \nabla \cdot \xi,$$

or

$$\partial_t u = \Delta u + \dot{W}, \quad u(0) = f,$$

where $\dot{W}$ is space-time white noise. This is the **stochastic heat equation**.

The SPDE reflects the interplay of macroscopic diffusion and microscopic fluctuation. In the next chapter, we will rigorously derive this limit.

6 Macroscopic Limits and SPDE Derivation

In this chapter, we establish the connection between the microscopic dynamics of the symmetric simple exclusion process (SSEP) and the macroscopic stochastic heat equation (SHE) that arises in the diffusive scaling limit. Starting from the definition of the fluctuation field, we derive a martingale decomposition, compute the quadratic variation, and identify the limiting process as a solution to an SPDE. The analysis culminates in a proof that the rescaled fluctuation field converges to a solution of the linear stochastic heat equation driven by space-time white noise.

6.1 Density Fluctuation Field and Diffusive Scaling

We define the fluctuation field associated to the SSEP on $\mathbb{Z}$ under diffusive scaling:

$$Y_t^\epsilon(\phi) := \sqrt{\epsilon} \sum_{x \in \mathbb{Z}} \phi(\epsilon x) \left(\eta_{\epsilon^{-2} t}(x) - \rho \right),$$

where $\rho \in (0, 1)$ is the density, $\phi \in \mathcal{S}(\mathbb{R})$, and $\epsilon \to 0$ is the scaling parameter.

The idea is to view Y_t^ϵ as a random distribution in $\mathcal{S}'(\mathbb{R})$, and study its limiting behavior as $\epsilon \to 0$. The generator of the SSEP leads to a discrete Laplacian acting on test functions, suggesting convergence to a solution of a stochastic PDE.

6.2 Martingale Decomposition

We apply Dynkin's formula to derive a decomposition into drift, martingale, and negligible terms.

[Martingale decomposition] For each $\phi \in \mathcal{S}(\mathbb{R})$,

$$Y_t^\epsilon(\phi) = Y_0^\epsilon(\phi) + \int_0^t Y_s^\epsilon(\Delta \phi)\, ds + M_t^\epsilon(\phi) + R_t^\epsilon(\phi),$$

where $M_t^\epsilon(\phi)$ is a martingale and $R_t^\epsilon(\phi) \to 0$ in probability.

Lemma 6.1 (Dynkin's formula)**.** *Let $f : \Omega \to \mathbb{R}$ be a local function. Then*

$$M_t^f := f(\eta_t) - f(\eta_0) - \int_0^t \mathcal{L} f(\eta_s)\, ds$$

is a martingale.

6.3 Quadratic Variation and Noise Identification

We compute the predictable quadratic variation $\langle M^\epsilon(\phi) \rangle_t$ to identify the noise strength.

Lemma 6.2. *Under the product measure ν_ρ, we have:*

$$\mathrm{Var}_{\nu_\rho}[\eta(x)] = \chi(\rho) = \rho(1 - \rho).$$

Proposition 6.3 (Quadratic variation limit)**.** *For $\phi \in \mathcal{S}(\mathbb{R})$,*

$$\langle M^\epsilon(\phi) \rangle_t \xrightarrow[\epsilon \to 0]{\mathbb{P}} 2\chi(\rho) t \int_{\mathbb{R}} (\phi'(x))^2 dx.$$

As a consequence, $M^\epsilon(\phi) \Rightarrow M(\phi)$, a continuous centered Gaussian martingale with variance $2\chi(\rho) t \int \phi^2$. Hence, the limit is a Wiener integral:

$$M_t(\phi) = \sqrt{2\chi(\rho)} \int_0^t \int_{\mathbb{R}} \phi(x) W(ds, dx).$$

6.4 SPDE Convergence and Limit Martingale Problem

We now describe the limiting behavior of the fluctuation field as $\epsilon \to 0$. We obtain the limiting decomposition:

$$Y_t(\phi) = Y_0(\phi) + \int_0^t Y_s(\Delta\phi)\, ds + M_t(\phi),$$

where $M_t(\phi)$ is a Gaussian martingale satisfying:

$$\langle M(\phi)\rangle_t = 2\chi(\rho)t \int \phi(x)^2 dx.$$

Definition 6.1 (Martingale solution / mild solution)**.** *A process $Y_t \in \mathcal{S}'(\mathbb{R})$ is a* martingale solution *to the stochastic heat equation if for all $\phi \in \mathcal{S}(\mathbb{R})$, the process*

$$M_t(\phi) := Y_t(\phi) - Y_0(\phi) - \int_0^t Y_s(\Delta\phi)\, ds$$

is a continuous martingale with quadratic variation

$$\langle M(\phi)\rangle_t = 2\chi(\rho)t \int \phi(x)^2 dx.$$

Equivalently, in mild form:

$$Y_t(\phi) = \langle Y_0, P_t\phi\rangle + \sqrt{2\chi(\rho)} \int_0^t \int_{\mathbb{R}} \nabla P_{t-s}\phi(x)\, W(ds, dx),$$

where P_t is the heat semigroup and W is space-time white noise.

Theorem 6.4 (Limit SPDE)**.** *Let Y_t^ϵ be the fluctuation field associated with the SSEP under diffusive scaling. Then:*

$$Y^\epsilon \Rightarrow Y \quad \text{in } D([0,T], \mathcal{S}'(\mathbb{R})),$$

where Y is the unique solution to the linear stochastic heat equation:

$$\partial_t Y = \Delta Y + \sqrt{2\chi(\rho)}\dot{W}.$$

Proof Sketch. The convergence is shown by:

- Proving tightness (by Kipnis–Landim criteria);
- Showing the limit satisfies the martingale problem;
- Uniqueness follows from Gaussianity.

□

7 Martingale Problem and Mild Formulation

This chapter aims to give a rigorous and self-contained formulation of the limiting fluctuation field Y arising from the symmetric simple exclusion process (SSEP). We begin by characterizing Y as the solution to a martingale problem associated with the linear stochastic heat equation (SHE). We then derive the precise structure of the noise term by computing its covariance and linking it to space-time white noise. These tools will also justify the equivalence of the martingale formulation and the mild (stochastic convolution) formulation of the solution. In particular, we show that both characterizations describe the same Gaussian process in $\mathcal{S}'(\mathbb{R})$, governed by the dynamics of the linear SHE. This duality offers complementary perspectives: the martingale formulation emphasizes probabilistic structure and convergence arguments, while the mild formulation connects the SPDE to semigroup theory and integral representations.

7.1 Martingale Problem Formulation

We recall from Chapter 5 that, for each test function $\phi \in \mathcal{S}(\mathbb{R})$, the fluctuation field $Y_t^\epsilon(\phi)$ satisfies the martingale decomposition

$$Y_t^\epsilon(\phi) = Y_0^\epsilon(\phi) + \int_0^t Y_s^\epsilon(\Delta\phi)\, ds + M_t^\epsilon(\phi) + R_t^\epsilon(\phi),$$

where the error term $R_t^\epsilon(\phi) \to 0$ in probability as $\epsilon \to 0$, and the martingale part $M_t^\epsilon(\phi)$ converges to a centered Gaussian martingale with known quadratic variation.

Definition 7.1 (Martingale solution to the linear SHE)**.** *A stochastic process* $Y = \{Y_t\}_{t\in[0,T]} \in \mathcal{S}'(\mathbb{R})$ *is said to solve the* martingale problem *if, for every test function* $\phi \in \mathcal{S}(\mathbb{R})$*, the process*

$$M_t(\phi) := Y_t(\phi) - Y_0(\phi) - \int_0^t Y_s(\Delta\phi) ds$$

is a continuous square-integrable martingale with quadratic variation

$$\langle M(\phi)\rangle_t = 2\chi(\rho)t \int \phi(x)^2 dx.$$

Proposition 7.1 (Uniqueness of martingale problem)**.** *Let* Y *and* $\tilde{Y}$ *solve the martingale problem with the same initial law. Then* $Y \overset{d}{=} \tilde{Y}$ *as* $\mathcal{S}'(\mathbb{R})$*-valued processes.*

Proof. Fix test functions $\phi_1, \ldots, \phi_n$. The processes $Y_t(\phi_i)$ and $\tilde{Y}_t(\phi_i)$ are continuous Gaussian martingales with identical mean and covariance:

$$\mathbb{E}[M_t(\phi_i)M_t(\phi_j)] = 2\chi(\rho)t \int \phi_i(x)\phi_j(x)\, dx.$$

Hence the finite-dimensional distributions match, implying $Y \overset{d}{=} \tilde{Y}$. □

7.2 Quadratic Variation and Noise Structure

[Covariance structure of the martingale] Let Y solve the martingale problem. Then for all $\phi, \psi \in \mathcal{S}(\mathbb{R})$,

$$\mathbb{E}[M_t(\phi)M_s(\psi)] = 2\chi(\rho)(t \wedge s) \int_{\mathbb{R}} \phi(x)\psi(x)dx.$$

Proof. This follows from the bilinearity of the covariation and polarization identity:

$$\langle M(\phi), M(\psi)\rangle_t = 2\chi(\rho)t \int \phi(x)\psi(x)\, dx.$$

□

This identifies $M_t(\phi)$ as a Wiener integral:

$$M_t(\phi) = \sqrt{2\chi(\rho)} \int_0^t \int_{\mathbb{R}} \phi(x) W(ds, dx),$$

where W is space-time white noise with covariance

$$\mathbb{E}[W([0,t] \times A)W([0,s] \times B)] = (t \wedge s)\lambda(A \cap B).$$

7.3 Mild Solution via Stochastic Convolution

Definition 7.2 (Mild solution to the SHE)**.** *Let P_t be the heat semigroup. A process $Y \in \mathcal{S}'(\mathbb{R})$ solves*

$$\partial_t Y = \Delta Y + \sqrt{2\chi(\rho)}\dot{W}$$

in the mild sense if

$$Y_t(\phi) = \langle Y_0, P_t\phi\rangle + \sqrt{2\chi(\rho)} \int_0^t \int_{\mathbb{R}} P_{t-s}\phi(x) W(ds, dx)$$

for all $\phi \in \mathcal{S}(\mathbb{R})$.

Theorem 7.2 (Equivalence of mild and martingale solutions)**.** *The mild and martingale formulations define the same process.*

Sketch. Use semigroup properties and Itô isometry to verify that the mild solution satisfies the martingale condition, with matching quadratic variation. □

7.4 Interpretation and Physical Meaning

Diffusion. The term ΔY corresponds to smoothing of density fluctuations, arising from local jumps in the underlying particle model.

Noise. The space-time white noise $\dot{W}$ accounts for microscopic randomness. Its coefficient $\sqrt{2\chi(\rho)}$ reflects the strength of density fluctuations in equilibrium.

Compressibility. The factor

$$\chi(\rho) = \mathrm{Var}_{\nu_\rho}[\eta(x)] = \rho(1-\rho)$$

is the *static compressibility*, indicating how likely particle occupation deviates from the mean.

Stochastic vs. Deterministic Limits. The empirical average obeys a deterministic law:

$$\partial_t \rho = \Delta \rho,$$

whereas the fluctuation field satisfies a stochastic correction:

$$\partial_t Y = \Delta Y + \sqrt{2\chi(\rho)}\dot{W}.$$

This contrast highlights the Gaussian fluctuations around the hydrodynamic limit.

8 Conclusion and Outlook

In this thesis, we have established a rigorous mathematical connection between the classical heat equation and its stochastic counterpart—the stochastic heat equation (SHE)—through the microscopic dynamics of the symmetric simple exclusion process (SSEP). The overall narrative demonstrates how tools from functional analysis, probability theory, and stochastic processes can be coherently combined to describe the emergence of macroscopic stochasticity from microscopic randomness.

8.1 From Classical to Stochastic: A Structural Summary

The classical heat equation

$$\partial_t u = \Delta u$$

describes the smooth evolution of a deterministic density profile over time. It arises naturally as the hydrodynamic limit of conservative particle systems like the SSEP, where the empirical density converges, under diffusive scaling, to a deterministic function solving this PDE.

However, when studying the fluctuations around this deterministic limit, we observe persistent randomness due to local particle interactions and stochastic transitions. The centered and properly rescaled fluctuation field

$$Y_t^\epsilon(\phi) := \sqrt{\epsilon} \sum_{x \in \mathbb{Z}} \phi(\epsilon x) \left(\eta_{\epsilon^{-2}t}(x) - \rho\right)$$

converges in distribution to a generalized solution Y of the stochastic heat equation:

$$\partial_t Y = \Delta Y + \sqrt{2\chi(\rho)}\dot{W},$$

where $\chi(\rho) = \rho(1-\rho)$ and $\dot{W}$ is space-time white noise.

This limit was analyzed rigorously via:

- Functional analytic tools (semigroup theory and mild solutions),
- Probabilistic techniques (martingale decomposition and central limit arguments),
- Structural identification (quadratic variation computations and SPDE convergence).

8.2 Scope and Limitations

The stochastic heat equation (SHE) describes the evolution of a continuously fluctuating field under the competing effects of diffusion and randomness. Intuitively, it models a medium where noise is injected locally and constantly, while diffusion spreads and smooths this noise over space and time. The result is a field that remains irregular and stochastic, yet exhibits spatial and temporal correlations shaped by the diffusive mechanism. This makes the SHE a fundamental model in the study of mesoscopic fluctuations in systems such as particle transport, heat conduction with disorder, or random walks in random environments.

The results presented in this thesis provide a rigorous derivation of the linear SHE from the symmetric simple exclusion process (SSEP), but the analysis is currently restricted to several idealized assumptions. First, the derivation assumes that the system is initialized in equilibrium, with particle configurations drawn from the Bernoulli product measure at fixed density. This equilibrium assumption simplifies the analysis by ensuring stationarity and symmetry, but limits the applicability of the results to nonequilibrium settings, where the density varies spatially or where boundary-driven effects play a role. Extending the framework to these settings requires deeper tools from the theory of nongradient systems and dynamical large deviations.

Second, the limiting stochastic PDE obtained in this work is linear:

$$\partial_t u(t,x) = \Delta u(t,x) + \dot{W}(t,x),$$

where $\dot{W}(t,x)$ denotes space-time white noise. This equation captures first-order Gaussian fluctuations and is appropriate when the underlying microscopic dynamics are symmetric. However, in many systems with weak asymmetry or external forcing, nonlinear effects become prominent. In such cases, the correct macroscopic limit is given by the Kardar–Parisi–Zhang (KPZ) equation:

$$\partial_t h(t,x) = \nu \Delta h(t,x) + \frac{\lambda}{2}(\nabla h(t,x))^2 + \dot{W}(t,x),$$

where $h(t,x)$ represents the height function of a growing interface, $\nu > 0$ is the viscosity (diffusion) coefficient, and $\lambda \neq 0$ controls the strength of the nonlinearity. The additional term $\frac{\lambda}{2}(\nabla h)^2$ breaks Gaussianity and introduces asymmetry into the growth dynamics.

Importantly, through the **Cole–Hopf transformation**, defined by

$$u(t,x) := e^{\frac{\lambda}{2\nu}h(t,x)},$$

the KPZ equation can be transformed into a multiplicative stochastic heat equation:

$$\partial_t u = \nu \Delta u + \frac{\lambda}{2}\dot{W}(t,x)u,$$

which establishes a formal connection between the SHE and KPZ universality class. While the linear SHE studied in this thesis belongs to the Gaussian universality class, the KPZ equation governs much richer, non-Gaussian, scale-invariant fluctuations.

Finally, this thesis focuses entirely on one-dimensional systems. In higher dimensions, the space-time white noise appearing in the SHE becomes too singular to interpret directly, and classical solution theories break down. Making sense of the SHE or KPZ equation in dimension two or higher typically requires more sophisticated techniques, such as the theory of regularity structures or energy solutions, which are beyond the scope of this thesis.

8.3 Outlook and Further Directions

The mathematical framework developed here lays the foundation for several promising research directions. One natural extension is to move beyond equilibrium and investigate fluctuations in systems with spatially varying initial data or subject to external driving. These nonequilibrium models may lead to stochastic PDEs with spatially inhomogeneous noise or even new fluctuation behaviors that cannot be captured by the classical SHE framework.

Another important direction is the study of nonlinear fluctuation fields. The KPZ equation mentioned above provides a canonical example of nonlinear stochastic dynamics with rich behavior, including non-Gaussian fluctuations, asymmetric spreading, and universal scaling limits. Connecting microscopic particle models with the KPZ equation through weak asymmetry scaling remains a central goal in the field, bridging probability, statistical mechanics, and mathematical physics.

More broadly, an ongoing question is that of universality: under what conditions do different microscopic models converge to the same macroscopic SPDE? Identifying such universality classes — whether for the linear SHE, the nonlinear KPZ, or other stochastic models — offers deep insights into the fundamental structures governing random systems.

Finally, from an applied and computational perspective, the stochastic PDEs studied in this thesis serve as effective models in various scientific contexts, including transport in random media, chemical reaction diffusion, and population dynamics. Developing numerical methods to simulate these equations accurately, and validating them against empirical data, is a fruitful direction for both theoretical and practical advancement.

In summary, while this thesis focuses on the linear stochastic heat equation in equilibrium and one dimension, it opens the door to a much wider landscape of questions about noise, diffusion, and universality in complex systems. These questions lie at the intersection of analysis, probability, and physics, and invite continued exploration both mathematically and through applications.

References