ON THE DIVISIBILITY BETWEEN CHARACTER DEGREES AND CODEGREES

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Abstract. Let G be a finite group and $\mathrm{Irr}(G)$ denote the irreducible characters of G. We define the codegree of $\chi \in \mathrm{Irr}(G)$ to be $\chi^c(1) = \frac{|G:\ker \chi|}{\chi(1)}$. Lately, mathematicians are studying the divisibility relationships between the degree and codegree of irreducible characters. This started with research by Liang and Qian, who defined and characterized $\mathscr{H}$-groups [10], which are groups with $\gcd(\chi(1), \chi^c(1)) = 1$ for every irreducible character χ. In another paper by Gao, Wang, and Chen [3], it was shown that there is no finite group G satisfying the condition that $\gcd(\chi(1), \chi^c(1))$ is prime for all $\chi \in \mathrm{Irr}(G)^{\#}$. Moreover, the author of this paper contributed to a result classifying the nonsolvable component of groups where $\gcd(\chi(1), \chi^c(1))$ is square-free for every $\chi \in \mathrm{Irr}(G)$, thereby solving an open problem posed by Guohua Qian [14]. The focus of this thesis is to present progress in a couple problems of this type. We further characterize groups satisfying the square-free character hypothesis by studying the structure of solvable cases. Additionally, we approach an entirely different open problem concerning groups where $\pi(\chi(1)) \subseteq \pi(\chi^c(1))$ for each $\chi \in \mathrm{Irr}(G)$. In particular, we show that various nonsolvable groups fail to meet this condition, suggesting that the hypothesis is a solvability condition.

The study of groups has been an endeavor dating back to the early 1830s, when Galois wrote his *Memoirs on the Conditions for Solvability of Equations by Radicals*. Galois would be the first to define normal subgroups and simple groups, discovering the Alt_n for $n > 4$ and $\mathrm{PSL}(2, p)$ for primes $p > 5$ as the first known nonabelian simple groups. In 1892, Otto Hölder proved that every nonabelian finite simple group has order a product of at least four (not necessarily distinct) primes. Additionally, Hölder called for the full classification of finite simple groups. Shortly after Hölder's theorem, William Burnside and Ferdinand Frobenius began studying the character theory of finite groups. The study of characters has been an indispensable tool for proving many celebrated results. The history of the Burnside theorem in particular demonstrates just how much easier groups can be studied using character-theoretic techniques. Although the result was originally proven in 1904, it wasn't until 1972 that a character-free proof was found [26][27]. Other landmark results on groups still have no proof that doesn't involve character theory, such as the Frobenius, Feit-Thompson, and Braurer-Suzuki theorems.

As one would expect, the proof of the classification of finite simple groups also involves a lot of character theory. Not only were the aformentioned theorems used in the classification, but also many of the autonomous parts of the long proof invoked character theoretic techniques. Most algebraists agree that the classification is practically impossible to prove otherwise.

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The significance of character theory is undeniable. In order to know as much as we do about groups, we had to depart from elementary techniques. Characters do just that by allowing us to study the properties of a group through linear algebra and ring theory. Moreover, the ordinary characters of a finite group allow for the construction of some very intriguing group invariants. It is for these reasons that characters themselves have become a topic of research. In this honors thesis, I offer a moderate introduction to character theory, a review of modern results concerning degrees and codegrees, as well as a few original results concerning a group's irreducible characters and their corresponding degrees and codegrees. In particular, we study the structure of groups satisfying some divisibility hypotheses between each irreducible character's degree and codegree.

Contents

1. Introduction to Character Theory

The study of character theory is rather technical. To a reader unfamiliar with the subject matter, a concise introduction to character theory will certainly help to understand the remainder of this paper. This first section is dedicated to a brief review of character theory largely motivated by Martin Isaacs's *Character Theory of Finite Groups* [5].

1.1. **Modules & Representations.**
Before we can appreciate what a character is, it's necessary that we develop an adequate background in the theory of algebras and modules.

Definition 1.1. Let $\mathbb{F}$ denote a field. If $\langle A, +\rangle$ is a vector space over $\mathbb{F}$ and $\langle A, +, \cdot\rangle$ is a ring with unity such that

$$(cx)y = c(xy) = x(cy)$$

for every $c \in \mathbb{F}$ and each $x, y \in A$, then we call A an **$\mathbb{F}$-algebra**. For our purposes, we always assume A is finite-dimensional.

Example 1.2. If $M_n(\mathbb{F})$ denotes the collection of all $n \times n$ matrices with entries in $\mathbb{F}$, then $M_n(\mathbb{F})$ is an $\mathbb{F}$-algebra with the usual matrix addition, matrix multiplication, and scalar multiplication.

Definition 1.3. If V is a vector space over $\mathbb{F}$, then we let $\operatorname{End}(V)$ denote the collection of linear transformations from V to itself.

It's not difficult to see that $\operatorname{End}(V)$ forms an $\mathbb{F}$-algebra with multiplication given by composition.

Definition 1.4. Given a finite group G and a field $\mathbb{F}$, we define the **group algebra** of G over $\mathbb{F}$ to be the set $\mathbb{F}[G]$ of all "linear combinations" of G. Every element of $\mathbb{F}[G]$ is written as $\sum_{g\in G} \lambda_g g$, where each λ_g is an element of $\mathbb{F}$.

As the name suggests, a group algebra is in fact an algebra. If $A = \mathbb{F}[G]$, we define addition by

$$\left(\sum_{g\in G} \lambda_g g\right) + \left(\sum_{g\in G} \delta_g g\right) = \sum_{g} (\lambda_g + \delta_g) g.$$

We also define scalar multiplication by

$$\gamma \left(\sum_{g\in G} \lambda_g g\right) = \sum_{g\in G} (\gamma \lambda_g) g.$$

Lastly, we define multiplication in A naturally. We can define multiplication by using our definition of addition. In particular, we define

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{g\in G}\delta_g g\right)=\sum_{h\in G}\left(\sum_{g\in G}\lambda_h\delta_g\right)(hg),$$

where the first sum indexed by h is our formal sum notation, and the sum indexed by g is the usual addition of scalars. As we will later see, the group algebra will be an important structure for further discussion of character theory.

Definition 1.5. Suppose that A is an $\mathbb{F}$-algebra and V is a finite-dimensional vector space over $\mathbb{F}$. Further suppose that for each $v \in V$ and $x \in A$, we associate some unique element $vx \in V$. If for all $x, y \in A$, $v, w \in V$, and $c \in \mathbb{F}$ we have

(1) $(v + w)x = vx + wx$,
(2) $v(x + y) = vx + vy$,
(3) $(vx)y = v(xy)$,
(4) $(cv)x = c(vx) = v(cx)$,
(5) $v1 = v$,

then we call V an **A-module**. Moreover, if $W \subseteq V$ is an A-invariant subspace, then we call W a **submodule** of V. For our purposes, we always assume an A-module is finite-dimensional.

Remark 1.6. Notice how if V is an A-module, then the map $T_x : V \to V$ given by $v \mapsto vx$ is linear for every $x \in A$.

Definition 1.7. If A and B are $\mathbb{F}$-algebras, then a map $\varphi : A \to B$ satisfying

(1) $\varphi(xy) = \varphi(x)\varphi(y)$,
(2) $\varphi(1) = 1$,
(3) φ is linear,

then we call φ an **algebra homomorphism**.

Remark 1.8. Suppose V is an A-module. Since each $x \in A$ forms a linear map $T_x \in \mathrm{End}(V)$, we now consider the map $\mathcal{H} : A \to \mathrm{End}(V)$ given by $x \mapsto T_x$. It's easy to verify that $\mathcal{H}$ is an algebra homomorphism. The image of this homomorphism is denoted by A_V, which we call the **image of A under V**.

Definition 1.9. If V and W are A-modules, then a linear map $\varphi : V \to W$ such that $\varphi(vx) = \varphi(v)x$ for every $v \in V$ and $x \in A$ is called an **A-homomorphism**. We let $\mathrm{Hom}_A(V, W)$ denote the collection of A-homomorphisms from V to W. We also denote $\mathrm{Hom}_A(V, V)$ by $\mathbb{E}_A(V)$.

Notice how $\mathrm{Hom}_A(V, W)$ forms a vector space over $\mathbb{F}$ while $\mathbb{E}_A(V)$ forms an $\mathbb{F}$-algebra.

Definition 1.10. Let V be some nonzero A-module. We call V **irreducible** if the only proper submodule is the zero module.

Lemma 1.11 (Schur's Lemma)**.** *If V and W are irreducible A-modules, then every $0 \neq \varphi \in \mathrm{Hom}_A(V, W)$ has an inverse in $\mathrm{Hom}_A(W, V)$.*

Proof. As we would expect, if $0 \neq \varphi \in \mathrm{Hom}_A(V, W)$, then $\ker \varphi$ is a submodule of V while $\mathrm{Im}\varphi$ is a submodule of W. If $\ker \varphi = V$, then $\varphi = 0$. Hence $\ker \varphi = 0$ and φ is an injection. Moreover, $\mathrm{Im}\varphi \neq 0$ since $\varphi \neq 0$, so of course $\mathrm{Im}\varphi = W$ and φ is surjective, hence invertible. It's not hard to see that φ^{-1} is also an A-homomorphism, completing our proof. □

It turns out that any module of the group algebra can almost always be decomposed into a direct sum of irreducible modules. In particular, we will show that this is the case for group algebras over a field of characteristic 0.

Definition 1.12. An A-module, V, is **completely reducible** if for every submodule $W \subseteq V$, there exists a complementary submodule U such that $V = W \oplus U$.

The converse of the following theorem is also true, but for our purposes, we only need the following direction.

Theorem 1.13 (Maschke's Theorem)**.** *Let G be a finite group and $\mathbb{F}$ be a field whose characteristic doesn't divide $|G|$. If V is any $\mathbb{F}[G]$-module, then V is completely reducible.*

Proof. Let W be any submodule. Further let U be a subspace of V such that $V = W \oplus U$. We then let $\theta : V \to W$ be the projection of V onto W by U. By this we mean that $\theta(v) = \theta(w+u) = w$ where $w \in W$ and $u \in U$. It follows that θ is a linear map. Define $\varphi : V \to W$ by

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} \theta(vg) g^{-1}.$$

Of course, φ is also linear. Notice how

$$\begin{aligned}\varphi(vh) &= \frac{1}{|G|} \sum_{g \in G} \theta(vhg) g^{-1} = \frac{1}{|G|} \sum_{g \in G} \theta(vhg) g^{-1} h^{-1} h = \frac{1}{|G|} \sum_{g \in G} \theta(vhg)(hg)^{-1} h \\ &= \frac{1}{|G|} \left(\sum_{g \in G} \theta(vhg)(hg)^{-1} \right) h = \frac{1}{|G|} \left(\sum_{g \in G} \theta(vg) g^{-1} \right) h = \varphi(v) h.\end{aligned}$$

So φ is an $\mathbb{F}[G]$-homomorphism. Moreover, since $wg \in W$ for every $w \in W$ and every $g \in G$, it follows that $\theta(wg) = wg$ and thus $\varphi(w) = w$. For any $v \in V$, we see that $\varphi(v) \in W$ so $\varphi(\varphi(v)) = \varphi(v)$. Hence $\varphi(v - \varphi(v)) = \varphi(v) - \varphi(v) = 0$, meaning $v - \varphi(v) \in \ker \varphi$.
By letting $\hat{U} = \ker \varphi$, we notice that $v = \varphi(v) + (v - \varphi(v)) \in W + \hat{U}$ for each $v \in V$.

Furthermore, $W \cap \hat{U} = \{0\}$ by construction, therefore $V = W \oplus \hat{U}$, where $\hat{U}$ is a submodule since it is the kernel of a homomorphism. □

Theorem 1.14. *If V is a finite-dimensional A-module, then V is completely reducible if and only if it is the direct sum of irreducible modules.*

Proof. First suppose that $V = \bigoplus_{\alpha \in I} V_\alpha$ for some index set I. Let $W \subseteq V$ be a submodule. Choose some maximal submodule $U \subseteq V$ such that $W \cap U = \{0\}$. If $W + U \neq V$, then we'd have that there exists some $V_\alpha \not\subseteq W + U$ and thus $W \cap (V_\alpha + U) = \{0\}$ contradicting maximality. Hence $W \oplus U = V$ and V is completely reducible.
Conversely, suppose V is completely reducible. Let $\mathcal{T}$ be the collection of all irreducible submodules of V and $W = \sum_{U \in \mathcal{T}} U$. If $W < V$, then we may choose some submodule $T \neq 0$ such that $W \oplus T = V$. Since T is also finite-dimensional, it must contain some irreducible submodule, in contadiction with the fact that $W \cap T = \{0\}$. Thus $W = V$ and V is necessarily the sum of some irreducible submodules. Now let $\hat{W}$ be a maximal submodule of V such that $\hat{W}$ is the direct sum of irreducible submodules in $\mathcal{T}$. We proceed once again by contradiction. If $\hat{W} < V$, then there must exist some $V' \in \mathcal{T}$ such that $V' \not\subseteq W$. By the irreducibility of V', it follows that $W \cap V' = \{0\}$ and that $\hat{W} \oplus V'$ properly contains $\hat{W}$, contradicting the maximality of $\hat{W}$. Thus $\hat{W} = V$ and V is the direct sum of irreducible modules. □

By a combination of this last theorem and Maschke's theorem, we arrive at the following corollary.

Corollary 1.15. *Suppose G is a finite group. If V is a finite-dimensional $\mathbb{C}[G]$-module, then V is the direct sum of irreducible $\mathbb{C}[G]$-modules.*

As we shall see, there are finitely many irreducible $\mathbb{C}[G]$-modules. Thus by knowing only those modules, we'll be able to characterize every $\mathbb{C}[G]$-module. We now finally discuss representations, although you will soon realize that they are the algebra's modules in disguise.

Definition 1.16. If A is an $\mathbb{F}$-algebra, a **representation** of A is an algebra homomorphism $\rho : A \to M_n(\mathbb{F})$. Moreover, we call n the **degree** of the representation ρ. Two representations of A, ρ and ϕ, are called **similar** if there exists some invertible $n \times n$ matrix E such that $\rho(a) = E\phi(a)E^{-1}$ for all $a \in A$.

Now suppose ρ is a representation of A. If V is the n-dimensional row-space, then $v\rho(a) \in V$ for every $v \in V$ and $a \in A$. Hence, we may define the A-module of V by the correspondence $va = v\rho(a)$. Conversely, if V is an A-module, then we may choose a basis $\mathcal{B}$ of V and let $\rho : A \to M_n(\mathbb{F})$ be given by $\rho(a) = T_a \in \mathrm{End}(V)$ where $T_a(v) = va$ for each $v \in V$. Here, we simply let T_a mean the matrix representation of the linear map with respect to $\mathcal{B}$. As we saw earlier, this is in fact an algebra homomorphism, although the representation may differ depending on the basis of V that you choose. As we should expect, starting with a representation ρ and creating an n-dimensional A-module as above, by choosing the right basis we can derive ρ back.

Proposition 1.17. *If $\rho, \phi : A \to M_n(\mathbb{F})$ are representations with corresponding A-modules V and W respectively, then ρ and ϕ are similar if and only if V and W are isomorphic.*

Proof. If ρ and ϕ are similar, then there exists some invertible $n \times n$ matrix E such that $\rho(a) = E\phi(a)E^{-1}$. We notice that E determines a linear transformation from V to W. Moreover, it's trivial to check that this map is in fact an algebra isomorphism. Conversely, suppose that V and W are isomorphic. Let $\mathcal{B}_V$ and $\mathcal{B}_W$ be the bases for which V determines ρ and W determines ϕ. For any $\psi \in \mathrm{Hom}_A(V, W)$, a matrix E is determined with respect to bases $\mathcal{B}_V$ and $\mathcal{B}_W$. In particular, let ψ be a module isomorphism. It follows that $\psi(va) = \psi(v)a$ for each $v \in V$ and $a \in A$. So $\rho(a)E = E\phi(a)$. Since ψ is an isomorphism, E is of course invertible, implying that $\rho(a) = E\phi(a)E^{-1}$. □

From this perspective, we may view the equivalence classes of similar representations to be in one-to-one correspondence with the equivalence classes of isomorphic modules.

Definition 1.18. A representation $\rho : A \to M_n(\mathbb{F})$ is called **irreducible** if the corresponding A-module is irreducible.

Notice how if V is an A-module and W is a proper submodule of V, then by choosing a basis $\mathcal{B}_W$ and an extension of this basis $\mathcal{B}_V$, then we may determine a representation ρ from V with respect to $\mathcal{B}_V$. By enumerating $\mathcal{B}_V$ such that the last $m = \dim W$ are the basis vectors of $\mathcal{B}_W$, then we have that

$$\rho(a) = \begin{bmatrix} \phi(a) & * \\ 0 & \zeta(a) \end{bmatrix}$$

where $\zeta : A \to M_m(\mathbb{F})$ and $\phi : A \to M_{n-m}(\mathbb{F})$ are representations of W and V/W respectively (here, V/W is defined as one would expect a factor module to be defined). If $V = W \oplus U$, then the top right block of $\rho(a)$ is also 0 and ϕ is also a representation of U. In other words, an irreducible representation could've been equivalently defined as a representation NOT similar to the above form. However, this definition avoids many pertinent details that make representations easier to study.

In the case that $A = \mathbb{C}[G]$ for some finite group G, then by Corollary 1.15, every irreducible representation is similar to a representation in some block diagonal form. Moreover, these blocks are irreducible representations.

Definition 1.19. The **general linear group** of dimension n over a field $\mathbb{F}$ is denoted $\mathrm{GL}(n, \mathbb{F})$ and is defined as the group of nonsingular $n \times n$ matrices with entries in $\mathbb{F}$. A group homomorphism $\phi : G \to \mathrm{GL}(n, \mathbb{F})$ is called a **group representation**.

Since the elements of a finite group G form a basis for the algebra $\mathbb{F}[G]$, it's easy to show that a group representation of G can be extended to a representation of $\mathbb{F}[G]$. Conversely, a representation of $\mathbb{F}[G]$ can be restricted to the elements of G to form a group representation. This follows since if $\rho : \mathbb{F}[G] \to M_n(\mathbb{F})$ is a representation, $\rho(1) = 1$ and thus $\rho(gg^{-1}) = \rho(g)\rho(g^{-1}) = 1$, meaning the image of $\rho|_G$ is contained in $\mathrm{GL}(n, \mathbb{F})$. Since there is a one-to-one correspondence between group representations and the usual representations, we will treat corresponding representations as though they are the same. We call a group representation **irreducible** or **similar** to another group representation when the corresponding representations of the group algebras satisfy those properties.

1.2. **The Basics of Characters.**

Definition 1.20. If $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ is a representation, then $\chi_\rho(g) = \mathrm{tr}\rho(g)$ is called the **character** afforded by ρ. We say χ_ρ is of **degree** n.

If $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ affords the character χ, then we see that $\chi(1) = \mathrm{tr}(I_n) = n$, i.e. $\chi(1)$ is the degree of χ. Another immediately apparent property of characters follows from the fact that for any two matrices A and B, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. This elementary property of linear algebra is very informative on characters through the following lemma.

Lemma 1.21. *Suppose that $\chi : G \to \mathbb{F}$ is a character of G over the field $\mathbb{F}$. If ρ is a representation that affords χ, and ψ is similar to ρ, then ψ also affords the character χ. Moreover, χ is constant on conjugacy classes.*

Proof. If $\psi = E\rho E^{-1}$, then $\mathrm{tr}\psi(g) = \mathrm{tr}(E\rho(g)E^{-1}) = \mathrm{tr}(EE^{-1}\rho(g)) = \mathrm{tr}\rho(g) = \chi(g)$. We again use the symmetry property to see that for any $g \in G$ and $h \in \mathrm{cl}(g)$, $\chi(h) = \chi(xgx^{-1}) = \chi(xx^{-1}g) = \chi(g)$. □

Just as well call a representation ρ irreducible, the character that ρ affords is also called **irreducible** whenever ρ is as well. We shall soon see that two representations over $\mathbb{C}$ afford the same character if and only if they are similar, hence a character χ is irreducible if and only if every representation that affords χ is irreducible. To show this, we must revisit the topic of algebras and modules.

Definition 1.22. Suppose V is a completely reducible A-module and M is an irreducible A-module. $M(V)$ denotes the sum of all submodules of V which are isomorphic to M, and we call $M(V)$ the M**-homogeneous** part of V.

As we will soon show, the structure of a completely reducible A-module can be best understood in terms of it's homogeneous parts.

Lemma 1.23. *Suppose* $V = \bigoplus_{i=1}^{n} W_i$ *where each* W_i *is an irreducible* A*-module. Moreover, let* M *be any irreducible* A*-module. It follows that*

- $M(V)$ *is an* $\mathbb{E}_A(V)$*-submodule of* V*;*
- $M(V) = \sum_{N \in \mathcal{C}} N$ *such that* $\mathcal{C} = \{W_i | W_i \cong M\}$*;*
- *If* $n_M(V)$ *denotes the number of* W_i *isomorphic to* M*, then it's invariant of the choice of* V*-decomposition.*

Proof. Let $\gamma \in \mathbb{E}_A(V)$ be arbitrary. Further suppose that $W \subseteq V$ is an A-submodule such that $W \cong M$. If $W \subseteq \ker \gamma$, then the inclusion $W\gamma \subseteq W$ follows trivially. Otherwise, when $W\gamma \neq \{0\}$, we have that $\psi = \gamma|_W \in \mathrm{Hom}_A(W, V)$. Since W is irreducible and $\ker \psi \neq W$, we must have that $\ker \psi = \{0\}$ and thus that $\psi(W) = W\gamma \cong M$. So of course, $W\gamma \subseteq M(V)$, and thus $M(V)\gamma \subseteq M(V)$ since every isomorphic copy of M satisfies this inclusion.
Now notice how $\sum_{N \in \mathcal{C}} N \subseteq M(V)$ trivially. For each $N \in \mathcal{C}$, let $\pi_N : V \to N$ be the projection map. For an A-submodule $W \subseteq V$ isomoprhic to M, $\pi_N(W) = 0$ or $\pi_N(W) = N$. In the latter case, $N \cong W$. □

Definition 1.24. If A is an $\mathbb{F}$-algebra, then A is itself an A-module by right multiplication. We call this the **regular** A-module and denote it by A°.
We call A a **semisimple** algebra whenever A° is completely reducible.

Notice how by Maaschke's theorem, $\mathbb{F}[G]$ is semisimple whenever $\mathrm{char}(\mathbb{F})$ doesn't divide $|G|$. As we shall soon see, every irreducible module of some algebra is "embedded" in the structure of the regular module. This is especially true when the algebra is semisimple.

Lemma 1.25. *If* A *is an* $\mathbb{F}$*-algebra, then every irreducible* A*-module is isomorphic to some factor module of* A°*.*

Proof. If V is an arbitrary irreducible A-module, then choose $0 \neq v \in V$. Let $\phi : A \to V$ be given by $\phi(x) = vx$ so that $\phi \in \mathrm{Hom}_A(A^\circ, V)$. Since $0 \neq v \in \mathrm{Im}\phi$, it follows by irreducibility that $\mathrm{Im}\phi = V$. Thus $V \cong A^\circ / \ker \phi$. □

Corollary 1.26. *If* A *is a semisimple* $\mathbb{F}$*-algebra, then every irreducible* A*-module is isomorphic to some submodule of* A°*.*

Proof. If V is an arbitrary irreducible A-module, then by Lemma 1.25, we have that $V \cong A^\circ/W$ for some submodule W of A°. By complete reducibility, $A^\circ = W \oplus U$ for some submodule U, and $A^\circ/W \cong U$. Hence $V \cong U \leq A^\circ$. □

The purpose of this lemma and corollary is to create a well-defined set of all irreducible modules of a given algebra (up to isomorphism). If A is an $\mathbb{F}$-algebra, then we let $\mathscr{M}(A)$ be a collection of irreducible factor modules of A° such that there exists exactly one copy of each irreducible module. Creating such a representative set is made possible by Lemma 1.25.

Corollary 1.27. *If V is any completely reducible A-module, and $\mathscr{M}(A)$ is our representative set of all irreducible modules, then $V = \bigoplus_{M\in\mathscr{M}(A)} M(V)$. More specifically, $A^\circ = \bigoplus_{M\in\mathscr{M}(A)} M(A^\circ)$.*

Now that we better understand the regular module and semisimple algebras, we can state and prove Wedderburn's theorem.

Theorem 1.28 (Wedderburn's Theorem)**.** *Let A be a semisimple algebra and M be an irreducible A-module. Then the following statements hold:*

- *If W is irreducible and distinct from M, then it is annihilated by $M(A^\circ)$.*
- *The map $\mathcal{H} : M(A^\circ) \to \mathrm{End}(M)$, given by $x \mapsto T_x : M \to M$, is injective with image $A_M \subseteq \mathrm{End}(M)$.*
- *$M(A^\circ)$ is a minimal ideal of A.*

Proof. Notice how the map $\rho_x : A^\circ \to A^\circ$ given by $a \mapsto xa$ is a homomorphism for each $x \in A^\circ$. In other words, $\rho_x \in \mathbb{E}_A(A^\circ)$ and by Lemma 1.23, we have that $xM(A^\circ) = \rho_x(M(A^\circ)) \subseteq M(A^\circ)$. Moreover, since $M(A^\circ)$ is a submodule of A°, it follows that $M(A^\circ)x \subseteq M(A^\circ)$, hence $M(A^\circ)$ is a two-sided ideal. Now if W is irreducible and distinct from M, then $W(A^\circ) \cap M(A^\circ) = \{0\}$ as a consequence of Lemma 1.23. Since both $W(A^\circ)$ and $M(A^\circ)$ are ideals, $W(A^\circ)M(A^\circ) = \{0\}$. By Corollary 1.26, there exists some submodule $\hat{W}$ of A° such that $\hat{W} \cong W$, thus

$$\hat{W} \subseteq W(A^\circ) \implies \hat{W}[M(A^\circ)] \subseteq W(A^\circ)M(A^\circ) = \{0\} \implies \hat{W}[M(A^\circ)] = \{0\}.$$

Since $W \cong \hat{W}$, they must have the same annihilators.
Let $S_x : W \to W$ be the map given by $w \mapsto wx$. Since W is annihilated by $M(A^\circ)$ for distinct irreducible modules, $S_x = 0$ for every $x \in M(A^\circ)$ whenever $M \ncong W$. Since $A = \bigoplus_{N\in\mathscr{M}(A)} N(A^\circ)$, it follows that if $x \in A$ and $m \in M(A^\circ)$ is the $M(A^\circ)$ component of x with respect to the decomposition, then $vx = vm$ and $T_x = T_m$. Thus the map $\mathcal{H} : M(A^\circ) \to \mathrm{End}(M)$ has image A_M as desired. Now suppose $x \in \ker\mathcal{H}$. In other words, x annihilates $M(A^\circ)$. But also, x annihilates every other irreducible A-module. It follows that x also annihilates A° since the regular module is completely reducible. Of course, $1x \in A^\circ x = \{0\}$, thus $x = 1x = 0$. Therefore $\ker\mathcal{H} = \{0\}$ and $\mathcal{H}$ is injective.
Suppose there exists some ideal $I < M(A^\circ)$ of A. There must exist some irreducible component $\hat{M}$ of $M(A^\circ)$ isomorphic to M such that $\hat{M} \nsubseteq I$. As such, $\hat{M} \cap I < \hat{M}$ which implies that $\hat{M} \cap I = \{0\}$ by irreducibility. But $\hat{M}I \subseteq I$ since I is an ideal and $\hat{M}I \subseteq \hat{M}$ since $\hat{M}$ is an A°-submodule. Thus $\hat{M}$ is annihilated by I, and by way of isomorphism, I also annihilates M. Hence for each $x \in I$, we see that $T_x = 0 \implies x \in \ker\mathcal{H}$ which must mean that $x = 0$. So $I = \{0\}$ and $M(A^\circ)$ is minimal. □

Wedderburn's theorem is usually stated differently. In particular, it states that a semisimple algebra is the direct sum of **simple** algebras, which are algebras containing no proper nontrivial ideals. This version of the theorem follows from Theorem 1.28 since every M-homogeneous part of a completely reducible regular module is annihilated by all the other homogeneous parts, thereby making any ideal of $M(A^\circ)$ an ideal of A. In other words, our study of semisimple algebras is completed by studying the simple algebras of the form A_M.

Lemma 1.29. *Suppose A is a semisimple $\mathbb{F}$-algebra and M is an irreducible A-module. If $E = \mathbb{E}_A(M)$, then $\mathbb{E}_E(M) = A_M$*

Proof. Since we may choose some $\hat{M} \cong M$ to be an irreducible component of A°, proving that the result is true for $\hat{M}$ implies that the result is true for M by way of isomorphism. Thus, we assume that $M \subseteq A^\circ$. If $T_x \in A_M$ and $\rho \in E$ are arbitrary, then we see that $T_x(v\rho) = T_x(\rho(v)) = \rho(v)x = \rho(vx) = \rho(T_x(v)) = T_x(v)\rho$. In other words, $T_x \in \mathbb{E}_E(M)$ and we see that $A_M \subseteq \mathbb{E}_E(M)$.
Conversely, let $\theta \in \mathbb{E}_E(M)$ be arbitrary. We notice that $(v\gamma)\theta = (v\theta)\gamma$ for $\gamma \in E$ and any $v \in M$. Define $\gamma_v : M \to A$ to be the map $m \mapsto vm$. We have that $\gamma_v(M) = vM \subseteq M$ since M is an ideal by Theorem 1.28, so $\gamma_v \in \mathrm{End}(M)$. If $a \in A$ and $m \in M$, then $\gamma_v(xa) = v(xa) = (vx)a = \gamma_v(x)a$ so that $\gamma_v \in E$ for each $v \in M$. Thus for each $m, v \in M$, $(vm)\theta = (m\gamma_v)\theta = (m\theta)\gamma_v = v(m\theta)$.
Now notice how $M(A^\circ)$ forms an algebra with multiplicative identity e, the component of 1 in the decomposition $A = \bigoplus_{N \in \mathscr{M}(A)} N(A)$. By fixing $0 \neq n \in M$, we find that $AnA \subseteq A(M(A^\circ))A \subseteq M(A^\circ)$. Since $M(A^\circ)$ is a minimal ideal, it follows that either $AnA = \{0\}$ or $AnA = M(A^\circ)$. Of course, $0 \neq n = 1 \cdot n \cdot 1 \in AnA$ so $AnA = M(A^\circ)$.
Now let $a, b \in A$ such that $anb = e$. Moreover, let $a = \sum_{N \in \mathscr{M}(A)} a_N$ and $b = \sum_{N \in \mathscr{M}(A)} b_N$ be the unique linear combinations of a and b such that $a_N, b_N \in N(A^\circ)$ for each $N \in \mathscr{M}(A)$. Since n annihilates any elements a_N and b_N for $N \ncong M$, we see that $e = a_M n b_M$. Notice how

$$m\theta = (m(a_M n b_M))\theta = ((ma_M)(nb_M))\theta = (ma_M)((nb_M)\theta) = m[a_M((nb_M)\theta)]$$

for every element $m \in M$. Thus $\theta = T_u \in A_M$ where $u = a_M((nb_M)\theta)$. □

Lemma 1.29 is precisely the kind of result we need to arrive at our most important theorem thus far.

Theorem 1.30. *Let A be a semisimple $\mathbb{F}$-algebra where $\mathbb{F}$ is algebraically closed. If M is an irreducible A-module, then*

- $A_M = \mathrm{End}(M)$;
- $\dim(A_M) = \dim(M(A^\circ)) = \dim(M)^2$;
- $n_M(A^\circ) = \dim(M)$;
- $\dim(\mathbf{Z}(A)) = |\mathscr{M}(A)|$.

Proof. Let $S = \{c \cdot 1_M \in \mathbb{E}_A(M) \mid c \in \mathbb{F}\}$ where 1_M is the identity map on M. For any $\theta \in \mathbb{E}_A(M)$, there must exist some eigenvalue $c \in \mathbb{F}$ of θ. It follows that $\theta - c \cdot 1_M \in \mathbb{E}_A(M)$. By Schur's lemma, we see that $\theta - c \cdot 1_M$ is either 0 or invertible. Of course, we may simply choose an eigenvector $v \in M$ of θ so that $(\theta - c \cdot 1_M)(v) = 0$, and so $\theta = c \cdot 1_M \in S$. In other words, $S = \mathbb{E}_A(M)$. By Lemma 1.29, we have that $\mathbb{E}_S(M) = A_M$. Of course, for any $\rho \in \mathrm{End}(M)$, we find that $\rho(v(c \cdot 1)) = \rho(cv) = c\rho(v) = \rho(v)(c \cdot 1)$, so that $\mathrm{End}(M) = \mathbb{E}_S(M) = A_M$.
Since $A_M = \mathrm{End}(M) \cong M_n(\mathbb{F})$ where $n = \dim(M)$, we see that $\dim(A_M) = \dim(M_n(\mathbb{F})) = \dim(M)^2$. Moreover, Wedderburn's theorem shows us that $A_M \cong M(A^\circ)$, hence $\dim(A_M) = \dim(M(A^\circ))$ as desired.
Using all our results thus far, we see that

$$\dim(M)^2 = \dim(M(A^\circ)) = \sum_{i=1}^{n_M(A^\circ)} \dim(M) = n_M(A^\circ)\dim(M).$$

So of course, $\dim(M) = n_M(A^\circ)$.
Lastly, notice how $\mathbf{Z}(A) = \bigoplus_{M \in \mathscr{M}(A)} \mathbf{Z}(M(A^\circ))$. Thus, it suffices to prove that $\dim(\mathbf{Z}(M(A^\circ))) = 1$ for each $M \in \mathscr{M}(A)$. Notice how for arbitrary $\rho \in \mathbb{E}_A(M)$ and $T_x \in A_M$, we have that $(\rho \circ T_x)(m) = \rho(mx) = \rho(m)x = (T_x \circ \rho)(m)$. In other words, $\mathbb{E}_A(M) \subseteq C_{\mathrm{End}(M)}(A_M)$, where $C_K(V)$ denotes the centralizer of V in K. Conversely, for every $\rho \in C_{\mathrm{End}(M)}(A_M)$ and $x \in A$, it follows that $\rho(mx) = (\rho \circ T_x)(m) = (T_x \circ \rho)(m) = \rho(m)x$. So $C_{\mathrm{End}(M)}(A_M) = \mathbb{E}_A(M)$. But as we've shown, $\mathbb{E}_A(M) = S \subseteq A_M$, so $\mathbf{Z}(M(A^\circ)) \cong \mathbf{Z}(A_M) = S$ and $\dim(\mathbf{Z}(M(A^\circ))) = 1$ since $\dim(S) = 1$. □

With this last theorem, we're finally able to prove a few more key results about the characters of a group. From now on, the set $\mathrm{Irr}(G)$ denotes the collection of all irreducible complex characters of G.

Proposition 1.31. *If ρ and ψ are nonsimilar complex representations of a group G, then they afford different characters. More specifically, the classes of irreducible representations afford distinct characters.*

Proof. Let M and N be the irreducible $\mathbb{C}[G]$-modules corresponding to ρ and ψ respectively. Moreover, let e_M and e_N be the $M(\mathbb{C}[G]^\circ)$ and $N(\mathbb{C}[G]^\circ)$ components of 1 by the decomposition $\bigoplus_{V \in \mathscr{M}(\mathbb{C}[G])} V(\mathbb{C}[G]^\circ)$. As per Wedderburn's theorem, we have that e_M annihilates N and e_N annihilates M. Thus $\rho(e_M) \neq 0$ and $\psi(e_N) \neq 0$ while $\rho(e_N) = 0 = \psi(e_M)$. If we let $\xi_\rho = \mathrm{tr}(\rho)$ and $\xi_\psi = \mathrm{tr}(\psi)$, then clearly $\xi_\rho|_G = \chi_\rho$ and $\xi_\psi|_G = \chi_\psi$. But since $\xi_\psi(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \xi_\psi(g) = \sum_{g \in G} a_g \chi_\psi(g)$ and similarly $\xi_\rho(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \chi_\rho(g)$, we see that the extension of the characters χ_ρ and χ_ψ to the entire group algebra is uniquely determined by the characters themselves, just as group representations uniquely determine the group algebra representations. But $\xi_\rho(e_N) \neq \xi_\psi(e_N)$, so χ_ρ and χ_ψ must be distinct. □

Notice how $\mathscr{M}(\mathbb{C}[G])$ is finite since $\mathbb{C}[G]^\circ$ is finite dimensional and thus can be decomposed into finitely many irreducible submodules. Moreover, by Lemma 1.21 and Proposition 1.17, we see that $|\mathscr{M}(\mathbb{C}[G])| = |\mathrm{Irr}(G)|$.

Corollary 1.32. *If G is a finite group, then*

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|.$$

Proof. By Theorem 1.30, $\dim(M(\mathbb{C}[G]^\circ)) = \dim(M)^2 = \chi(1)^2$ where $M \in \mathscr{M}(\mathbb{C}[G])$ and χ is the character afforded by the representations corresponding to M. Thus

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = \sum_{M \in \mathscr{M}(\mathbb{C}[G])} \dim(M(\mathbb{C}[G]^\circ)) = \dim\left(\bigoplus_{M \in \mathscr{M}(\mathbb{C}[G])} M(\mathbb{C}[G]^\circ)\right) = \dim(\mathbb{C}[G]).$$

The Corollary follows from the fact that G is a basis of $\mathbb{C}[G]$, ergo $|G| = \dim(\mathbb{C}[G])$. $\square$

Although we already know that there are finitely many irreducible characters for each group G, it'd be helpful if the exact number of irreducible characters can be determined by the properties of G alone. Fortunately, this next theorem does exactly that, giving us a pleasant equality.

Theorem 1.33. *Let $\mathcal{J}_1, ..., \mathcal{J}_t$ denote the conjugacy classes of G and $K_i = \sum_{x \in J_i} x \in \mathbb{C}[G]$. The set $\{K_1, ...K_t\}$ forms a basis for $\mathbf{Z}(\mathbb{C}[G])$. Moreover, if $K_iK_j = \sum_{r=1}^t a_{i,j,r}K_r$, then the coefficients $a_{i,j,r}$ are nonnegative integers.*
Additionally, $|\mathrm{Irr}(G)|$ is equal to the number of conjugacy classes in G.

Proof. Of course, for arbitrary $g \in G$, we have that

$$gK_ig^{-1} = g\left(\sum_{x \in J_i} x\right)g^{-1} = \sum_{x \in J_i} gxg^{-1} = \sum_{x \in J_i} x = K_i.$$

It follows that for any $y \in \mathbb{C}[G]$, we have that $yK_iy^{-1} = K_i \implies yK_i = K_iy$. In other words, $K_i \in \mathbf{Z}(\mathbb{C}[G])$ for each $i = 1, ..., t$. Moreover, the K_i are linearly independent since for any K_i and K_j distinct, the elements they sum over are disjoint. Now let $\sum_{g \in G} a_g g \in \mathbf{Z}(\mathbb{C}[G])$ be arbitrary. If $h \in G$, then $h(\sum_{g \in G} a_g g)h^{-1} = \sum_{g \in G} a_g hgh^{-1} = \sum_{g \in G} a_g g$. So we must have that $a_g = a_{hgh^{-1}}$ for each $g \in G$, and thus the coefficients are constant on the conjugacy classes of G. In other words, $\sum_{g \in G} a_g g \in \mathrm{span}(K_1, ..., K_t)$ and $\{K_1, ..., K_t\}$ forms a basis of $\mathbf{Z}(\mathbb{C}[G])$.
From the definition of multiplication on a group algebra, we have that $a_{i,j,t} = |\{(x, y) \in \mathcal{J}_i \times \mathcal{J}_j \mid xy = g\}|$ for any $g \in \mathcal{J}_t$. Thus each of the coefficients of K_iK_j are nonnegative integers.
From these results, $|\mathrm{Irr}(G)| = \dim(\mathbf{Z}(\mathbb{C}[G])) = t$ follows immediately. $\square$

Corollary 1.34. *A finite group G is abelian if and only if every irreducible character is **linear**, i.e. of degree* 1.

Proof. Notice how G is abelian if and only if the number of conjugacy classes, t is equal to $|G|$. Of course, $\chi(1) \geq 1$ for each $\chi \in \mathrm{Irr}(G)$. If any $\chi \in \mathrm{Irr}(G)$ were nonlinear, then $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 > \sum_{\chi \in \mathrm{Irr}(G)} 1 = |G|$ where the first and last equalities follow from Corollary 1.31 and Theorem 1.32. □

1.3. **Class Functions and Orthogonality.**

Definition 1.35. Given a finite group G, a **class function** of G is a map $\phi : G \to \mathbb{C}$ that is constant on the conjugacy classes of G.

By Lemma 1.21, it should be clear that every character of G is also a class function. Though the relationship between characters and class functions is quite a bit more profound than you'd expect.

Theorem 1.36. *If V denotes the collection of all class functions on a group G, then V forms a vector space over $\mathbb{C}$ with basis $\mathrm{Irr}(G)$. Moreover, if $\phi \in V$ is nonzero and such that $\phi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$, then ϕ is a character of G if and only if each a_χ is a nonnegative integer.*

Proof. The fact that V forms a vector space is elementary to prove. We also see that $\dim(V)$ is equal to the number of conjugacy classes on G, ergo $\dim(V) = |\mathrm{Irr}(G)|$. Thus it suffices to show that $\mathrm{Irr}(G)$ is linearly independent. If $\sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi = 0$, then we shold have that $\sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi(1) = 0$. But since $\chi(1) > 0$ for each $\chi \in \mathrm{Irr}(G)$, we must have that $a_\chi = 0$ for all χ.
We first note that the characters of G are closed under addition. If χ and λ are characters afforded by the representations ρ and ψ, then

$$\phi(g) = \begin{bmatrix} \rho(g) & 0 \\ 0 & \psi(g) \end{bmatrix}$$

is another group representation that affords the character $\chi + \lambda$. Hence, all linear combinations of irreducible characters, with nonnegative integers as coefficients, are also characters. Conversely, if λ is a character afforded by the $\mathbb{C}[G]$-module V, then we can decompose V into it's irreducible components so that $\lambda = \sum_{\chi \in \mathrm{Irr}(G)} n_{M_\chi}(V)\chi$, where $M_\chi \in \mathscr{M}(\mathbb{C}[G])$ is the irreducible module corresponding to χ. □

Although we've shown that the irreducible representations determine distinct characters, we can say even more about characters as they relate to representations.

Corollary 1.37. *If ρ and ψ are $\mathbb{C}$-representations of the group G, then ρ and ψ are similar if and only if they afford the same character.*

Proof. The fact that similar representations afford the same character is merely a repetition of Lemma 1.21. In the converse direction, let V and W be the $\mathbb{C}[G]$-modules corresponding to ρ and ψ. We must have that $\chi_\rho = \sum_{\chi \in \mathrm{Irr}(G)} n_{M_\chi}(V)\chi$ and $\chi_\psi = \sum_{\chi \in \mathrm{Irr}(G)} n_{M_\chi}(W)\chi$. Since $\mathrm{Irr}(G)$ forms a basis for the class function space, $\chi_\rho = \chi_\psi$ would imply that $n_{M_\chi}(V) = n_{M_\chi}(W)$ for each $\chi \in \mathrm{Irr}(G)$. Thus, $W \cong V$ which means ρ and ψ are similar by Proposition 1.17. □

Example 1.38. Let Sym_3 denote the symmetric group on 3 elements. Since Sym_3 has 3 conjugacy classes and 6 elements, we must have that $\mathrm{Irr}(\mathrm{Sym}_3) = \{\chi_1, \chi_2, \chi_3\}$ such that $\chi_1(1)^2 + \chi_2(1)^2 + \chi_3(1)^2 = 6$. Since one character must be the trivial character, it follows that the only other possible irreducible character degrees are 1 and 2 respectively.
In order to detail the irreducible characters in a more concise fashion, algebraists created **character tables**. The character table of a group G has rows corresponding to the elements of $\mathrm{Irr}(G)$ and columns corresponding to the conjugacy classes of G. Using what we know, we can fill in some of the entries of the character table for Sym_3 below. Here, $\mathrm{Cl}(g)$ denotes the conjugacy class of $g \in G$.

$\mathrm{Cl}_G(g)$	1	$(1,2,3)$	$(1,2)$
χ_1	1	1	1
χ_2	1	a_1	a_2
χ_3	2	a_3	a_4

We'll soon discover character relations that help us fill out the remaining four entries of the table. Though it is at this point that we can appreciate how much easier it is to fill out a character table than it is to construct the irreducible representations of a group.

Definition 1.39. If ψ is an arbitrary character of G, then we say $\lambda \in \mathrm{Irr}(G)$ is an **irreducible constituent** of ψ when $\psi = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \chi$ is such that $n_\chi > 0$.
We also define the character $\rho = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$ to be the **regular character** of G.

This next lemma might convince you that the regular character is redundant, but rest assured, we will find it useful in establishing the *character orthogonality relations*. These are precisely the sorts of theorems we need to help construct character tables, and we will return to the example of Sym_3.

Lemma 1.40. *For any $1 \neq g \in G$, $\rho(g) = 0$ where ρ is the regular character of G.*

Proof. Of course, ρ corresponds to the entire $\mathbb{C}[G]^\circ$ module by Theorem 1.30. Let $\phi : \mathbb{C}[G] \to M_n(\mathbb{C})$ be the corresponding representation given by the basis $G = \{g_1, ..., g_{|G|}\}$. Notice how if $\phi(g) = A$, then $A_{i,j} = 1$ when $g_i g = T_g(g_i) = g_j$. Otherwise, $A_{i,j} = 0$ since $T_g(g_i) \in G$. Hence, $\rho(g) = |\{h \in G \mid hg = h\}| = 0$. □

Theorem 1.41. *Recall how we let e_M denote the $M(\mathbb{C}[G]^\circ)$ component of 1 with respect to the decomposition $\bigoplus_{M\in\mathscr{M}(\mathbb{C}[G])} M(\mathbb{C}[G]^\circ)$. If χ_M is the irreducible character corresponding to $M \in \mathscr{M}(\mathbb{C}[G])$, then*

$$e_M = \frac{1}{|G|}\sum_{g\in G}\chi_M(1)\chi_M(g^{-1})g.$$

Proof. By Lemma 1.40, it follows that for each $g \in G$, $\rho(e_M g^{-1}) = a_g|G|$ where $e_M = \sum_{g\in G} a_g g$. This is because $\rho(e_M g^{-1}) = \rho([\sum_{g\in G} a_g g]g^{-1}) = \sum_{g\in G} a_g \rho(gg^{-1}) = a_g\rho(1) = a_g|G|$. By the definition of ρ, we have that

$$a_g|G| = \rho\left(e_M g^{-1}\right) = \sum_{\chi\in \mathrm{Irr}(G)}\chi(1)\chi(e_M g^{-1}).$$

But the element e_M annihilates M, hence $\chi(e_M) = 0$ whenever χ is not afforded by M. In other words, $\chi_N(e_M g^{-1}) = \chi(g^{-1})\delta_{N,M}$ where $\delta_{N,M}$ is the Kronecker delta function, i.e. $\delta_{N,M} = 1$ when $N \cong M$ and $\delta_{N,M} = 0$ otherwise. So $a_g|G| = \chi_M(1)\chi_M(g^{-1})$ for each $\chi \in \mathrm{Irr}(G)$. Taking a formal sum indexed by G, we derive

$$|G|e_M = |G|\sum_{g\in G} a_g g = \sum_{g\in G}\chi_M(1)\chi_M(g^{-1})g,$$

and the desired result follows by dividing a factor of $|G|$. □

Theorem 1.42 (First Orthogonality Relation)**.** *For every $h \in G$, we have that*

$$\frac{1}{|G|}\sum_{g\in G}\chi_M(gh)\chi_N(g^{-1}) = \delta_{N,M}\frac{\chi_M(h)}{\chi_M(1)}.$$

In particular, when we let $h = 1$, we find

$$\frac{1}{|G|}\sum_{g\in G}\chi_M(g)\chi_N(g^{-1}) = \delta_{N,M}.$$

Proof. Since $e_M e_N = 0$ whenever $M \ncong N$, we notice that

$$e_M = e_M \cdot 1 = e_M \sum_{N\in\mathscr{M}(\mathbb{C}[G])} N(\mathbb{C}[G]^\circ) = e_M^2.$$

As a result, $e_M e_N = e_M\delta_{M,N}$. So, by Theorem 1.41, we derive that

$$e_M e_N = \frac{1}{|G|^2}\sum_{h\in G}\sum_{g\in G}\chi_M(1)\chi_M(h^{-1})\chi_N(1)\chi_N(g^{-1})hg$$

$$= \frac{\chi_M(1)\chi_N(1)}{|G|^2}\sum_{h\in G}\sum_{g\in G}\chi_M(gh^{-1})\chi_N(g^{-1})h = \frac{\delta_{M,N}}{|G|}\sum_{h\in G}\chi_M(1)\chi_M(h^{-1})h = e_M\delta_{M,N}.$$

Hence, $\chi_M(1)\chi_N(1)\sum_{g\in G}\chi_M(gh^{-1})\chi_N(g^{-1}) = \delta_{M,N}|G|\chi_M(1)\chi_M(h^{-1})$ for each $h \in G$. By substituting h for h^{-1}, the relations follow. □

As the name of Theorem 1.42 suggests, this isn't the only orthogonality relation of characters.

Lemma 1.43. *Suppose ϕ is the representation of G that affords the character χ. If $g \in G$ and $n = o(g)$, then*

- *$\phi(g)$ is similar to a diagonal matrix A;*
- *$A_{i,i}^n = 1$ for each $i = 1, ..., d = \chi(1)$;*
- *$\chi(g) = A_{1,1} + ... + A_{d,d}$ and $|\chi(g)| \leq \chi(1)$;*
- *$\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof. By letting $\theta = \phi|_{\langle g \rangle}$, we define a representation on $\langle g \rangle$. By Maschke's theorem, we see that θ is similar to a block-diagonal form representation. Assume that θ is already of this form. Since $\langle g \rangle$ is abelian, the irreducible blocks of θ are linear representations. Hence $\theta(g) = A$ for some diagonal matrix. Since $1 = \theta(1) = \theta(g^n) = \theta(g)^n = A^n$, we have that $A_{i,i}^n = 1$.
Of course, $\chi(g) = \text{tr}(\theta(g)) = A_{1,1} + ... + A_{d,d}$. Moreover, since $A_{i,i}^n = 1$, we see that $|A_{i,i}| = 1$. Thus $|\chi(g)| \leq \chi(1)$ by the triangle inequality. Furthermore, $\theta(g^{-1}) = B$ a diagonal matrix such that $B_{i,i} = A_{i,i}^{-1}$. Because each $|A_{i,i}| = |B_{i,i}| = 1$, we can conclude that $B_{i,i} = \overline{A_{i,i}}$ and thus that $\chi(g^{-1}) = \overline{\chi(g)}$. □

Corollary 1.44 (First Orthogonality Relation Reformulated)**.**

$$\frac{1}{|G|} \sum_{g \in G} \chi_M(g)\overline{\chi_N(g)} = \delta_{i,j}$$

The first orthogonality relation is so important to character theory, that we turn the space of class functions into a finite dimensional Hilbert space by endowing it with an inner product motivated by the relation.

Definition 1.45. If φ_1 and φ_2 are class functions on G, then the **inner product** of φ_1 and φ_2 is denoted and defined by

$$[\varphi_1, \varphi_2] = \frac{1}{|G|} \sum_{g \in G} \varphi_1(g)\overline{\varphi_2(g)}.$$

It's not hard to see that the previous definition does define an inner product. It is in fact the case that

- $[\varphi, \vartheta] = \overline{[\vartheta, \varphi]}$;
- $[c\varphi_1 + d\varphi_2, \vartheta] = c[\varphi_1, \vartheta] + d[\varphi_2, \vartheta]$;
- $[\vartheta, c\varphi_1 + d\varphi_2] = \overline{c}[\vartheta, \varphi_1] + \overline{d}[\vartheta, \varphi_2]$;
- and $[\varphi, \varphi] \geq 0$, with equality if and only if $\varphi = 0$.

Moreover, as we've verified in Corollary 1.44, $\text{Irr}(G)$ forms an orthonormal basis for the space of all G class functions. In other words, if $[\varphi, \chi_M] = c_M$ for each $M \in \mathscr{M}(\mathbb{C}[G])$, then $\varphi = \sum_{M \in \mathscr{M}(\mathbb{C}[G])} c_M \chi_M$. We use this fact to establish the following theorem.

Theorem 1.46. *If ψ and φ are characters of G, then $[\psi,\varphi]=[\varphi,\psi]$ and it's always a nonnegative integer. Moreover, $\psi\in\operatorname{Irr}(G)$ if and only if $[\psi,\psi]=1$.*

Proof. If $\chi\in\operatorname{Irr}(G)$, then $[\chi,\chi]=1$ since it's a component of itself exactly once. Now let $\psi=\sum_{M\in\mathscr{M}(\mathbb{C}[G])}a_M\chi_M$ and $\varphi=\sum_{M\in\mathscr{M}(\mathbb{C}[G])}b_M\chi_M$. By linearity in the first argument and conjugate linearity in the second, we see that

$$[\psi,\varphi]=\sum_{M\in\mathscr{M}(\mathbb{C}[G])}a_M[\chi_M,\varphi]=\sum_{M,N\in\mathscr{M}(\mathbb{C}[G])}a_Mb_N[\chi_M,\chi_N]=\sum_{M\in\mathscr{M}(\mathbb{C}[G])}a_Mb_M[\chi_M,\chi_M]$$
$$=\sum_{M\in\mathscr{M}(\mathbb{C}[G])}a_Mb_M,$$

which is also equal to $[\varphi,\psi]$. Now if $\sum_{M\in\mathscr{M}(\mathbb{C}[G])}a_M^2=[\psi,\psi]=1$, then that must mean that $a_M=1$ for some $M\in\mathscr{M}(\mathbb{C}[G])$ and $a_N=0$ for all other irreducible modules. By Corollary 1.37, this must mean that ψ is irreducible. □

Now we proceed with the second orthogonality relation.

Theorem 1.47 (Second Orthogonality Relation)**.** *For $g,h\in G$ such that $h\notin\operatorname{Cl}(g)$,*

$$\sum_{\chi\in\operatorname{Irr}(G)}\chi(g)\overline{\chi(h)}=0.$$

If g and h are conjugates, then the sum is equal to $C_G(g)$.

Proof. If $\{g_M\}_{M\in\mathscr{M}(\mathbb{C}[G])}$ denotes a collection of representatives for each conjugacy class in G, then define $A_{M,N}=\chi_M(g_N)$. By ordering the $\mathscr{M}(\mathbb{C}[G])$, the entries of $A_{M,N}$ define a matrix A. Now we let D be the diagonal matrix with entries $\delta_{M,N}|\operatorname{Cl}(g_M)|$. By Corollary 1.44, we have that

$$|G|\delta_{M,N}=\sum_{g\in G}\chi_M(g)\overline{\chi_N(g)}=\sum_{V\in\mathscr{M}(\mathbb{C}[G])}|\operatorname{Cl}(g_V)|\chi_M(g_V)\overline{\chi_N(g_V)}.$$

By construction, we have that $|G|I=AD\overline{A}^T=D\overline{A}^TA$. Thus,

$$|G|\delta_{M,N}=\sum_{V\in\mathscr{M}(\mathbb{C}[G])}|\operatorname{Cl}(g_M)|\chi_V(g_M)\overline{\chi_V(g_N)}=\sum_{\chi\in\operatorname{Irr}(G)}|\operatorname{Cl}(g_M)|\overline{\chi(g_M)}\chi(g_N).$$

Since $|C_G(g)|=|G|/|\operatorname{Cl}(g)|$, our desired result follows. □

Example 1.48. Returning back to the character table of Sym_3, we notice how by Theorem 1.42,

$$0=\chi_1(1)\chi_2(1)+\chi_1((1\ 2\ 3))\chi_2((1\ 3\ 2))+\chi_1((1\ 3\ 2))\chi_2((1\ 2\ 3))+\chi_1((1\ 2))\chi_2((1\ 2))$$
$$+\chi_1((1\ 3))\chi_2((1\ 3))+\chi_1((2\ 3))\chi_2((2\ 3))=1+2a_1+3a_2.$$

Thus $2a_1+3a_2=-1$. Similarly, $2a_3+3a_4=-2$. Now applying Theorem 1.47, we also find that $1+a_1+2a_3=0=1+a_2+2a_4$ when we choose the identity for h. The character table can now be found deduced from a system of linear equations.

$\mathrm{Cl}_G(g)$	1	$(1,2,3)$	$(1,2)$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Lemma 1.49. *If χ is a character of G afforded by representation ϕ, then $\chi(1) = \chi(g)$ if and only if $g \in \ker\phi$.*

Proof. Clearly $\chi(g) = \chi(1)$ whenever $g \in \ker\phi$, so it suffices to prove the converse direction. By Lemma 1.43, $\chi(g) = A_{1,1} + ... + A_{n,n}$ where $n = \chi(1)$ and each $A_{i,i}$ is an $o(g)$ root of unity. But since $|A_{i,i}| = 1$, $\chi(g) = \chi(1)$ if and only if each $A_{i,i} = 1$. Thus, $\phi(g)$ is similar to $A = I$, and thus $\phi(g) = I$. □

In light of this lemma, we let $\ker\chi = \ker\phi$ where ϕ affords χ.

Theorem 1.50. *If ψ is a character of G with $\psi = \sum_{\chi\in\mathrm{Irr}(G)} n_\chi\chi$, then $\ker\psi = \bigcap_{\chi\in S}\ker\chi$ such that $S = \{\chi \in \mathrm{Irr}(G) \mid n_\chi = [\psi,\chi] > 0\}$.*

Proof. Of course, $\bigcap_{\chi\in S}\ker\chi \subseteq \ker\psi$ since $\psi(g) = \sum_{\chi\in S} n_\chi\chi(g) = \sum_{\chi\in S} n_\chi\chi(1) = \psi(1)$ for each $g \in \bigcap_{\chi\in S}\ker\chi$.
Conversely, suppose $g \in \ker\psi$. So $\sum_{\chi\in S} n_\chi\chi(g) = \psi(g) = \psi(1) = \sum_{\chi\in S} n_\chi\chi(1)$ and thus $\sum_{\chi\in S} n_\chi\chi(1) = |\sum_{\chi\in S} n_\chi\chi(g)| \leq \sum_{\chi\in S} n_\chi|\chi(g)|$. But Lemma 1.43 states that $|\chi(g)| \leq \chi(1)$, thus $\sum_{\chi\in S} n_\chi\chi(1) = \sum_{\chi\in S} n_\chi|\chi(g)|$ forcing each $\chi \in S$ to be such that $\chi(g) = \chi(1)$. □

Suppose χ is a character of G and that $N \lhd G$. If $N \leq \ker\chi$, then of course χ is constant on the cosets of $\ker\chi$ and thus the cosets of N. As it turns out, ψ is not only a class function, but also a character of G/N. We call ψ the **inflation** of χ to G/N and denote it by $\mathrm{Inf}_G^{G/N}\chi$.

Theorem 1.51. *If $N \lhd G$, then for a character χ of G with $N \leq \ker\chi$ $\psi = \mathrm{Inf}_G^{G/N}$, ψ is a character of G/N. Moreover, $\chi \in \mathrm{Irr}(G)$ if and only if $\psi \in \mathrm{Irr}(G/N)$.*

Proof. Let ϕ be the representation that affords χ. Since $N \leq \ker\chi$, we see that $\phi(g)$ is constant on the cosets of N. Let $\hat{\phi}$ denote the map $gN \mapsto \phi(g)$. Of course, $\hat{\phi}$ is also a representation of G/N and affords ψ. If ϕ is irreducible, then $\hat{\phi}$ must also be irreducible. Otherwise, it's similar to a block-diagonal form representation and thus so is ϕ, contradicting it's irreducibility. The same argument can be made in the converse direction. □

In light of this, we may view $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G)$ (this is of course an abuse of notation since the domain of maps in $\mathrm{Irr}(G/N)$ and the domain of maps in $\mathrm{Irr}(G)$ are distinct).

2. Character Degrees and Codegrees

So far, we've only discussed what characters of groups are and showcased how they are significantly easier to derive than the group representations. With as much exposition as we've gone through, we do have the tools necessary to prove theorems such as Burnside's theorem.

Theorem 2.1 ([26][27], Burnside's Theorem)**.** *If G is a group of order $p^a q^b$ for primes p and q and positive integers a and b, then G is solvable.*

Although results like these are interesting and historically connected to character theory, they are not the subject matter of this paper. Instead, the focus is to study the character degrees and codegrees as they relate to a group's structure. Of course, character degrees were known to us as soon as characters were created. On the other hand, character codegrees have only just been recently defined and studied within the last two decades. Although new, character codegrees are naturally defined in light of their neat properties.

2.1. **Modern Results on Character Degrees.** Before we review modern results, we mention a few neat properties that the degrees of irreducible characters satisfy.

Theorem 2.2 ([5], Theorem 3.11)**.** *If G is a finite group and $\chi \in \mathrm{Irr}(G)$, then $\chi(1)$ divides the order of G.*

The proof of this theorem follows from orthogonality relation described in Theorem 1.42 as well as the fact that the values of χ are algebraic integers. To help maintain the brevity of this thesis, we opt not to review a proof, as it would require a review of the properties that algebraic integers satisfy.

Theorem 2.3 ([5], Theorem 6.2, Clifford's Theorem)**.** *Suppose $N \lhd G$ and that $\chi \in \mathrm{Irr}(G)$. Moreover, let $\psi \in \mathrm{Irr}(N)$ be an irreducible constituent of $\chi|_N$. Then*

- *Every irreducible constituent $\vartheta \in \mathrm{Irr}(N)$ of $\chi|_N$ is a conjugate of ψ, in the sense that $\vartheta(g) = \psi^h(g) = \psi(hgh^{-1})$ for some $h \in G$;*
- *$\chi|_N = [\chi|_N, \psi] \sum_{i=1}^{s} \psi_i$ where the ψ_i are all the distinct conjugates of ψ;*
- *$\psi(1)$ divides $\chi(1)$, and in particular $\chi(1) = [\chi|_N, \psi] s\psi(1)$, where s is the number of distinct ψ conjugates over G.*

The proof of Clifford's theorem would require that we define the induced character and derive a few relevant properties. Once again, we opt to merely state the theorem in order to keep the contents of this thesis concise. One consequence of Clifford's theorem (as well as the existence and properties of an induced character), is the following corollary. By writing $N \lhd \lhd G$, I mean that N is **subnormal** in G. That is to say that there exists $N_0, ..., N_k$ such that $N = N_0 \lhd N_1 \lhd ... \lhd N_k = G$.

Corollary 2.4. *If $N \lhd \lhd G$ and $\psi \in \mathrm{Irr}(N)$, then ψ is an irreducible constituent $\chi|_N$ for some $\chi \in \mathrm{Irr}(G)$, hence $\psi(1) \mid \chi(1)$.*

Proof. We proceed by induction on the terms of the subnormal series $N \lhd N_1 \lhd \ldots \lhd N_k = G$. The result follows trivially on N_1 by a direct application of Clifford's theorem. If N_i also satisfies the induction hypothesis, then let $\vartheta \in \mathrm{Irr}(N_i)$ such that ψ is an irreducible constituent of $\vartheta|_N$. By applying Clifford's theorem to ϑ, we see that there exists some $\chi \in \mathrm{Irr}(G)$ such that ϑ is an irreducible constituent of $\chi|_{N_i}$. Thus $\vartheta|_N = \sum_{\lambda \in \mathrm{Irr}(N)} e_\lambda \lambda$ and $\chi|_{N_i} = \sum_{\lambda \in \mathrm{Irr}(N_i)} d_\lambda \lambda = d_\vartheta \vartheta + \sum_{\lambda \in \mathrm{Irr}(N_i) - \{\vartheta\}} d_\lambda \lambda$ where $d_\vartheta > 0$, hence $\chi|_N = d_\vartheta \vartheta|_N + \sum_{\lambda \in \mathrm{Irr}(N_i) - \{\vartheta\}} d_\lambda \lambda|_N$, implying that $[\chi|_N, \psi] > 0$ and thus that ψ is a constituent of $\chi|_N$. ☐

Definition 2.5. If G is a finite group, then the **character degree** set of G is denoted and defined by $\mathrm{c.\,d.}(G) = \{\chi(1) \in \mathbb{Z}^+ \mid \chi \in \mathrm{Irr}(G)\}$.

Properties such as Corollary 2.4 will eventually enable us to reduce problems concerning character degrees to cases of groups that are far more reasonable to understand.

Example 2.6. Suppose we wanted to check which groups satisfy the property that every element of $\mathrm{c.\,d.}(G)$ is a prime power. We call $\chi \in \mathrm{Irr}(G)$ a DP-character if $\chi(1)$ is a prime power, and we call G a DP-group if every $\chi \in \mathrm{Irr}(G)$ is a DP-character. Corollary 2.4 tells us that the normal subgroups of G must also satisfy this property. Thus, the contrapositive tells us that in order to check whether G fails this hypothesis, it suffices to prove that the minimal normal subgroups of G fail the property. Olaf Manz has already classified precisely which nonsolvable groups are DP-groups [11]. We provide a less powerful statement that details which nonabelian simple groups can be appear as the chief factors for a nonsolvable group G. This should hopefully illustrate the techniques used in the line of research.

Lemma 2.7 ([11])**.** *The nonabelian simple DP-groups are precisely* Alt_5 *and* $\mathrm{PSL}(2,8)$.

Proposition 2.8. *If G is a nonsolvable DP-group, then every nonabelian chief factor in the composition series of G is isomorphic to either* Alt_5 *or* $\mathrm{PSL}(2,8)$.

Proof. Suppose $\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ is a composition series of G and that G_{i+1}/G_i is a nonabelian simple factor such that $G_{i+1}/G_i \ncong \mathrm{Alt}_5$ and $G_{i+1}/G_i \ncong \mathrm{PSL}(2,8)$. By Lemma 2.6, there exists some $\psi \in \mathrm{Irr}(G_{i+1}/G_i)$ such that $\psi(1)$ is not a prime power. Since ψ can be viewed as a character on all of G_{i+1} by Theorem 1.51, we see that G_{i+1} is also not a DP-group. As we've already explained, the property of being a DP-group passes down to subnormal subgroups of G by Corollary 2.4. Thus G is either not a DP-group or has nonabelian chief factors all isomorphic to one of either Alt_5 or $\mathrm{PSL}(2,8)$. ☐

The statement and proof of Proposition 2.7 should serve as an instructive example on how to deal with character degree hypotheses. We should always check to see if the hypothesis passes down to normal/subnormal subgroups as well as quotient groups whenever possible.

The study of character degrees began roughly after Martin Isaacs and Donald Passman published their paper titled *A Characterization of Groups in Terms of the Degrees of Their Characters* in 1965 [6]. Since then, this group invariant has been studied thoroughly, yielding various fascinating and useful results. Perhaps the most surprising result is that of Itô-Michler.

Let $\pi(n)$ denote the collection of all prime divisors of $n \in \mathbb{Z}^+$. We write $\pi(G)$ to mean $\pi(n)$. Moreover, let $\rho(G) = \{p \in \pi(G) \mid \exists \chi \in \mathrm{Irr}(G) \text{ s.t. } p \mid \chi(1)\}$.

Theorem 2.9 ([7], Itô-Michler Theorem)**.** *If G a finite group then G has a nontrivial normal abelian Sylow-p subgroup if and only if $p \in \pi(G) - \rho(G)$, i.e. p is a prime dividing $|G|$ that doesn't divide any element of* $\mathrm{c.d.}(G)$*.*

Another important theorem concerning character degrees is Thompson's theorem, which gives a condition for the existence of a normal p-complement.

Theorem 2.10 ([5], Corollary 12.2, Thompson's Theorem)**.** *If p is a prime such that $p \mid \chi(1)$ for every nonlinear $\chi \in \mathrm{Irr}(G)$, then there exists some $N \lhd G$ such that $\gcd(|N|, p) = 1$ and $|G : N| = p^a$ for some $a \in \mathbb{Z}^+$.*

Of the existing theorems on character degrees, one that I find particularly interesting can be derived from the Broline-Garrison theorem on character kernels. Here and throughout the remainder of the paper, we let $\mathbf{F}(G)$ denote the **Fitting** subgroup of G, which is the maximal normal nilpotent subgroup of G.

Theorem 2.11 ([5], Theorem 12.19, Broline-Garrison Theorem)**.** *Let $\chi \in \mathrm{Irr}(G)$ be nontrivial. If $\ker \chi \nsubseteq \mathbf{F}(G)$ or $\ker \chi = \mathbf{F}(G)$ while $G/\ker \chi$ is solvable, then there must exist some $\psi \in \mathrm{Irr}(G)$ such that $\psi(1) > \chi(1)$ and $\ker \psi < \ker \chi$.*

Using the Broline-Garrison theorem, we can achieve a bound on the **Fitting length** of G, which is just the minimal group series length of G such that each factor is nilpotent. Equivalently, the Fitting length is the length of the **Fitting series**, where $1 \lhd F_1 \lhd ... \lhd F_n = G$, $F_1 = \mathbf{F}(G)$, and $F_{i+1}/F_i = \mathbf{F}(G/F_i)$ for each $i = 1, ..., n-1$. One can think of Fitting length as describing how close a group is to being nilpotent. We denote the Fitting length of G by $h(G)$. This next result was originally derived by Duane Garrison.

Theorem 2.12. *Suppose G is a solvable finite group. Then G has a Fitting series and we have that $h(G) \leq |\mathrm{c.d.}(G)|$.*

Proof. The fact that G has a Fitting series follows from the fact that all abelian groups are nilpotent. Since G is solvable, it must have a derived series, which is thus a group series with nilpotent factors.
We proceed to prove the upper bound on $h(G)$ by induction on $|G|$. Notice how $\mathrm{c.d.}(G/\mathbf{F}(G)) \subseteq \mathrm{c.d.}(G)$. By the induction hypothesis, $h(G/\mathbf{F}(G)) \leq |\mathrm{c.d.}(G/\mathbf{F}(G))|$. Let $\chi \in \mathrm{Irr}(G)$ be of maximal degree. If $\ker \chi \nless \mathbf{F}(G)$, then there must exist

some $\psi \in \mathrm{Irr}(G)$ with strictly greater degree. By contradiction, $\ker \chi < \mathbf{F}(G)$, and thus $\chi(1) \notin \mathrm{c.d.}(G/\mathbf{F}(G))$. Since $h(G) = h(G/\mathbf{F}(G)) + 1 \leq |\mathrm{c.d.}(G/\mathbf{F}(G))| + 1 \leq |\mathrm{c.d.}(G)|$, we see that the theorem follows. □

What makes this theorem particularly interesting is it's useful nature. Given a large group with enough "complexity" (such as groups with high rank), directly computing the Fitting length can be computationally expensive. In contrast, computing the character table of a group can be done much more efficiently as we saw in Example 1.38 and 1.48. Using well established theorems, we've devised algorithms executed by Computer Algebra Systems to compute the character table. Although only a bound, Theorem 2.7 showcases how by even just knowing as little information as the dimensions of irreducible representations, we have corollaries that describe the structure of the relevant group that could've been otherwise difficult to calculate.

Another bound exists in terms of the number of character degrees, but this time for the derived length.

Theorem 2.13 ([8], Kildetoft's Theorem)**.** *If G is a solvable finite group with $|\mathrm{c.d.}(G)| \geq 3$, then $d(G) \leq 2|\mathrm{c.d.}(G)| - 3$ where $d(G)$ denotes the derived length of G. Moreover, $d(G) \leq |\mathrm{c.d.}(G)|$ if $|\mathrm{c.d.}(G)| \leq 3$.*

A conjecture still remains open for a sharper bound, as both Martin Isaacs and Gary Seitz have asked for.

Conjecture 2.14. *For a solvable group G, we have that $d(G) \leq |\mathrm{c.d.}(G)|$.*

Similar to the case of computing the exact fitting length, it can take a lot of resources to calculate the derived length given enough size and complexity.

There is a myriad of other results connecting $\mathrm{c.d.}(G)$ to the structure of G. Hopefully by now you're convinced that the character degrees of a group are very informative and worth studying. Because of the fascinating connections character degrees have to various structural properties, character theorists would eventually search for similar group invariants. This would motivate the definition of a character codegree, which is seen as the dual value to an irreducible character's degree. As we will see, some theorems and properties of codegrees bare a striking similarity to results concerning character degrees, which should hopefully convince you that they too are worth studying.

2.2. **Modern Results on Character Codegrees.**

As an analogue to character degrees, the codegree is defined as follows.

Definition 2.15. For a character $\chi \in \mathrm{Irr}(G)$, we say that the **codegree** of χ is denoted and defined as

$$\chi^c(1) = \frac{|G : \ker(\chi)|}{\chi(1)}.$$

Moreover, we let $\mathrm{cod}(G) = \{\chi^c(1) \in \mathbb{Z}^+ \mid \chi \in \mathrm{Irr}(G)\}$.

Although this definition might seem arbitrary at first glance, it's actually much cleaner than we'd initially expect.

Lemma 2.16 ([5], Corollary 11.29)**.** *If $N \lhd G$ and $\chi \in \mathrm{Irr}(G)$, then an irreducible constituent $\psi \in \mathrm{Irr}(N)$ of $\chi|_M$ is such that $\chi(1)$ divides $|G : N|\psi(1)$.*

Lemma 2.17 ([5], Theorem 6.11)**.** *If $N \lhd G$ and $\psi \in \mathrm{Irr}(N)$ is an irreducible constituent of $\chi|_N$ for $\chi \in \mathrm{Irr}(G)$, then $\chi|_N = \psi$ whenever $G = N \ker \chi$.*

Theorem 2.18. *Suppose $N \lhd G$ and that $\chi \in \mathrm{Irr}(G)$ such that $N \subseteq \ker \chi$. Then χ can be viewed as an irreducible character of G/N with matching degree and codegree. Additionally, if $N \lhd \lhd G$ with $\chi \in \mathrm{Irr}(G)$ and $\psi \in \mathrm{Irr}(N)$, such that ψ an irreducible constituent of $\chi|_M$, then $\psi^c(1)$ divides $\chi^c(1)$.*

Proof. Suppose first that $N \subseteq \ker \chi$. By Theorem 1.51, we can take $\psi = \mathrm{Inf}_G^{G/N} \chi \in \mathrm{Irr}(G/N)$, which of course is such that $\psi(1) = \chi(1)$. Thus

$$\chi^c(1) = \frac{|G : \ker \chi|}{\chi(1)} = \frac{|G|}{\chi(1)|\ker \chi|} = \frac{|G|/|N|}{\chi(1)|\ker \chi|/|N|} = \frac{|G : N|}{\psi(1)|\ker \chi : N|}$$

$$= \frac{|G/N : \ker \psi|}{\psi(1)} = \psi^c(1).$$

Here, the fact that $|G||N|/|\ker \chi||N| = |\ker \psi|$ follows from the fact the triviality that $|\ker \psi| = |\ker \chi|/|N|$.
Now suppose $N = N_0 \lhd \ldots \lhd N_k = G$. We proceed by induction on the length of this subnormal series, hence it suffices to assume that $k = 1$ and that $N \lhd G$. Better yet, we may assume that N is maximal normal in G since we can induct on the composition series of G/N. Theorem 1.50 guarantees us that $\ker \chi \cap N = \ker \chi|_N \leq \ker \psi$. In the case that $\ker \chi \leq \ker \psi$, the result follow immediately from Lemma 2.16. Otherwise, when $\ker \chi \not\leq \ker \psi$, we see that $N \ker \chi = G$ by maximality. So

$$\psi^c(1) = \frac{|N : \ker \psi|}{\psi(1)} = \frac{|N : \ker \chi \cap M|}{\psi(1)} = \frac{|G : \ker \chi|}{\chi(1)} = \chi^c(1),$$

where $\psi(1) = \chi(1)$ follows by Lemma 2.17. □

The character codegree first appeared in a paper by Guohua Qian, et. al. titled *Co-degrees of irreducible characters in finite groups* in 2007 [19]. A growing body of research has shown that codegrees offer a novel yet just as powerful group invariant as character degrees do. First of all, it's worth mentioning that an analog of Itô-Michler doesn't exist for character codegrees due to the following theorem.

Theorem 2.19 ([14], Proposition 1.2)**.** *Every prime divisor p of $|G|$ is such that $p \mid \chi^c(1)$ for some $\chi \in \mathrm{Irr}(G)$.*

Although it might have been pleasantly surprising for us to establish a codegree analog of Itô-Michler, it's reassuring to know that codegrees are substantively different from degrees and can provide us with new information that $\mathrm{c.d.}(G)$ can't. Perhaps one of the explanations for Theorem 2.19 is that the following conjecture is true.

Conjecture 2.20 (Qian's Conjecture)**.** *For every element $g \in G$, there exists some $m \in \mathrm{cod}(G)$ such that $o(g)$ divides m.*

Originally formulated in 2011 [15], Qian's conjecture would be an incredibly surprising result if it were true. The current progress on this conjecture has proven its truthhood in the following cases:

- G is solvable [14];
- G is almost simple with nonabelian $\mathrm{Soc}(G)$, i.e. $S \lhd G \leq \mathrm{Aut}(S)$ for some nonabelian simple S [16];
- and G is Fitting-free, i.e. $\mathbf{F}(G) = \{1\}$ [17].

A few other partial results have been provided for certain element orders. For example,

Theorem 2.21 ([18], Main Result)**.** *If $g \in G$ has square-free element order, then $o(g) \mid \chi^c(1)$ for some $\chi \in \mathrm{Irr}(G)$.*

Moreover, a remark in the paper by Zeinab Akhlaghi, et. al. [17] states that for a minimal counterexample to Qian's conjecture, the socle $\mathrm{Soc}(G)$ must be solvable. It's not hard to see that $\mathrm{Soc}(G)$ must therefore be the direct product of elementary abelian groups. This is all to say that the structure of a counterexample is highly restricted, and that the conjecture is likely true.

Beyond Qian's conjecture and all its partial results, character codegrees are similar to character degrees in the sense that a codegree analog exists for the Broline-Garrison theorem.

Theorem 2.22 ([20], Theorem 1.1)**.** *If $\chi \in \mathrm{Irr}(G)$ is such that $\ker\chi \not\subseteq \mathbf{F}(G)$ or $\ker\chi = \mathbf{F}(G) < \mathrm{Sol}(G)$, then there exists some $\psi \in \mathrm{Irr}(G)$ such that $\ker\psi < \ker\chi$ and $\psi^c(1) > \chi^c(1)$.*

As is the case with character degrees in Theorem 2.12, Theorem 2.22 helps us prove a Fitting length bound. It's important to note that we consider the Fitting length of $\{1\}$ to be 0.

Theorem 2.23. *If G is a solvable group, then $h(G) \leq |\mathrm{cod}(G)| - 1$.*

Proof. The trivial character on G clearly has codegree 1. If G is nontrivial, then Theorem 2.19 tells us that $|\mathrm{cod}(G)| \geq 2$. In other words, we can assume that $G > \mathbf{F}(G)$. By letting $\chi \in \mathrm{Irr}(G)$ have maximal codegree, we see that $\ker\chi < \mathbf{F}(G)$ by Theorem 2.22. Thus $\chi^c(1) \notin \mathrm{cod}(G/\mathbf{F}(G))$ and $\mathrm{cod}(G/\mathbf{F}(G)) + 1 \leq \mathrm{cod}(G)$. The proof of the theorem follows by induction on the factors of the Fitting series, where the induction hypothesis tells us that

$$h(G) - 1 = h(G/\mathbf{F}(G)) \leq |\mathrm{cod}(G/\mathbf{F}(G))| - 1 \leq |\mathrm{cod}(G)| - 2,$$

and thus $h(G) \leq |\mathrm{cod}(G)| - 1$. □

Although this bound is already interesting, other Fitting length bounds have been established in terms of $|\mathrm{cod}(G)|$ that are far sharper than the bound in Theorem 2.23 for groups with many distinct character codegrees.

Theorem 2.24 ([20], Proposition 1.3 & Proposition 1.4)**.** *For a solvable group G,*

- $h(G) \leq \frac{1}{2}|\mathrm{cod}(G)| + 1$*;*
- $h(G) \leq 8\log_2(|\mathrm{cod}(G)|) + 80$*.*

Current research on codegrees includes studying the divisibility relationship between the degree and codegree of given irreducible characters. In particular, this relationship was originally studied in [10] when Liang and Qian defined and characterized the $\mathscr{H}$-groups up to isomorphism. An **$\mathscr{H}$-group** is a group G such that $\gcd(\chi(1), \chi^c(1)) = 1$ for every $\chi \in \mathrm{Irr}(G)$.

Theorem 2.25 ([10], Theorem A)**.** *If G is a solvable group, then it's an $\mathscr{H}$-group if and only if it satisifies one of the following conditions:*

(1) G is an abelian group.
(2) $G = M \ltimes \mathbf{F}(G)$ such that M is cyclic and $\mathbf{F}(G)$ is an abelian Hall subgroup of G where every $P \in \mathrm{Syl}_p(M)$ acts Frobeniusly on $[\mathbf{F}(G), P]$ for every prime p.
(3) $G = L \ltimes \mathbf{F}_2(G)$ while satisfying all the following:
  - *L is cyclic with square-free order and $\gcd(|L|, |\mathbf{F}_2(G) : \mathbf{F}(G)|) = 1$.*
  - *$\mathbf{F}_2(G)$ satisfies the condition described in (2).*
  - *If $p \in \pi(\gcd(|L|, |\mathbf{F}(G)|))$, then $O_{p'}(G)$ is a 2-Frobenius group (i.e., there exists $N, M \lhd O_{p'}(G)$ such that $O_{p'}(G)/N$ and M are both Frobenius with kernels M/N and N) such that $|O_{p'}(G) : M| = p$, $|N| = p^{ep}$, and $|M : N| = (p^{ep} - 1)/(p^e - 1)$ for some $e \in \mathbb{Z}^+$.*

Let $S(G)$ be the minimal normal subgroup of a group G such that $G/S(G)$ is solvable. Moreover, we denote and define the collection $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

Theorem 2.26 ([10], Theorem B)**.** *If G is a nonsolvable group, then it's an $\mathscr{H}$-group if and only if it satisfies all the following conditions:*

(1) $G = D \ltimes (L \times N)$ where D, L, and N are all Hall subgroups of G.
(2) $L = S(G)$ such that $L/\mathrm{Sol}(L) \cong \mathrm{PSL}(2, 2^f)$ for some integer $f \geq 2$ and either $\mathrm{Sol}(L) = 1$ or $|\mathrm{Sol}(G)| = 2^{2f}$ such that every $\chi \in \mathrm{Irr}(L) - \mathrm{Irr}(L/\mathrm{Sol}(L))$ has degree $2^{2f} - 1$.
(3) D is cyclic with $|D|$ square-free and dividing f.
(4) G/L is a solvable $\mathscr{H}$-group.

The classification of $\mathscr{H}$-groups motivated the study of groups satisfying similar conditions. Two examples include the following theorems established by Li Gao, Zhongbi Wang, and Guiyun Chen.

Theorem 2.27 ([3], Theorem 1.1)**.** *There does not exist a finite group G such that for every $1_G \neq \chi \in \mathrm{Irr}(G)$, $\gcd(\chi(1), \chi^c(1))$ is prime.*

Theorem 2.28 ([3], Theorem 1.2)**.** *If G is a finite group such that $\gcd(\chi(1), \chi^c(1))$ is prime for every $\chi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$, then G is solvable.*

Later, we will talk about an open problem posed by Guohua Qian about groups satisfying the property that $\gcd(\chi(1), \chi^c(1))$ is square-free for each $\chi \in \mathrm{Irr}(G)$.

One celebrated theorem, established by Stephen Gagola Jr. and Mark Lewis in 1999, characterizes nilpotent groups through a divisibility condition between character degrees and codegrees.

Theorem 2.29 ([21], Theorem A, Gagola-Lewis Theorem)**.** *If G is a finite group, then G is nilpotent if and only if $\chi(1)$ divides $\chi^c(1)$ for every $\chi \in \mathrm{Irr}(G)$.*

Remark 2.30. The Gagola-Lewis theorem was originally published in 1999, years before character codegrees were ever defined. The original statement of the theorem is that G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker \chi|$ for each $\chi \in \mathrm{Irr}(G)$. The fact that character theorists were interacting with character codegrees nearly a decade before they'd be formally defined should convince the reader that character codegrees are rather natural.

A generalization of Gagola-Lewis was established in 2017, where instead of checking that every irreducible character of G has degree dividing codegree, it suffices to check only some of $\mathrm{Irr}(G)$. A **monolithic character** $\chi \in \mathrm{Irr}(G)$ is a character such that $G/\ker \chi$ has a unique minimal normal subgroup. We denote the collection of all irreducible monolithic characters of G by $\mathrm{Irr}_m(G)$.

Theorem 2.31 ([22], Theorem 1.2)**.** *If G is a finite group, then G is nilpotent if and only if $\chi(1)$ divides $\chi^c(1)$ for every $\chi \in \mathrm{Irr}_m(G)$.*

In a later section, we will study a hypothesis weaker than that of Gagola-Lewis. We'll partially answer another open problem posed by Guohua Qian, namely Problem 5.2 in Qian's codegree survey article titled *Character codegrees in finite groups* [14]. In particular, we study groups G such that $\pi(\chi(1)) \subseteq \pi(\chi^c(1))$ for each $\chi \in \mathrm{Irr}(G)$. Equivalently, these are groups where for each prime p and $\chi \in \mathrm{Irr}(G)$, p divides $\chi^c(1)$ whenever p divides $\chi(1)$. The goal is to prove that this hypothesis is a solvability condition. We'll start by checking whether the nonabelian simple groups fails this hypothesis before working on proving that every almost simple group with nonabelian socle fails the hypothesis.

3. The Square-Free Character Hypothesis

This section is dedicated to Problem 5.7 in Guohua Qian's *Character codegrees in finite groups* [14]. The reader should notice how a solution to the problem generalizes Theorems 2.25, 2.26, and 2.27.

Problem: Let G be a finite group such that $\gcd(\chi(1), \chi^c(1))$ is square-free for every $\chi \in \mathrm{Irr}(G)$. If G is nonsolvable, what is the structure of $G/\mathrm{Sol}(G)$? If G is solvable, does there exist an upper bound on the Fitting length of G?

From now on, we say that such a group satisfies the **square-free character hypothesis**. The author of this thesis coauthored a paper that solved the nonsolvable case [23]. We first provide an exposition of the proof of the nonsolvable theorem. We then talk about a structural restriction in the solvable case. In particular, we provide equivalent conditions for a nilpotent group G satisfying the square-free character hypothesis.

3.1. **A Solution to the Nonsolvable Case.**

Corollary 3.1. *If $N \lhd \lhd G$ and $\psi \in \mathrm{Irr}(N)$, then there exists some $\chi \in \mathrm{Irr}(G)$ such that $\psi(1)$ divides $\chi(1)$ and $\psi^c(1)$ divides $\chi^c(1)$.*

The above corollary follows almost immediately from Corollary 2.4 and Theorem 2.18. We notice that if G is a group satisfying the square-free character hypothesis, then so does every subnormal subgroup of G. Moreover, it follows from Theorem 2.18 that for any $N \lhd G$ the quotient group G/N must also satisfy this condition. So the square-free character hypothesis is always inherited by normal subgroups and quotients. We'll rely on this fact heavily in the remainder of our proof.

This next lemma will help us prove that $G/\mathrm{Sol}(G)$ is necessarily an almost simple group with nonabelian socle.

Lemma 3.2 (Mattarei's Lemma)**.** *Let G be a finite group with $\mathrm{Sol}(G)$. Then there are non-abelian simple groups $S_1, \dots, S_r$ and positive integers $n_1, \dots, n_r \geq 1$ such that G contains an isomorphic copy of $T_1^{n_1} \times \cdots \times T_r^{n_r}$ and is isomorphic to a subgroup of $\mathrm{Aut}(S_1^{n_1}) \times \cdots \times \mathrm{Aut}(S_r^{n_r})$.*

Proof. We first show that every minimal normal subgroup of G is isomorphic to some S^k where S is nonabelian simple. Let M be a minimal normal subgroup of G and N be a minimal normal subgroup of M. Since $M \lhd G$, it follows that for any $g \in G$, gNg^{-1} is again a minimal normal subgroup of M. Let $N_1, ..., N_k$ be all the conjugates of N and $H = N_1 N_2 ... N_k$. For any $h \in H$ and $g \in G$, we see that $ghg^{-1} = gn_1n_2...n_kg^{-1} = gn_1g^{-1}gn_2g^{-1}...gn_kg^{-1}$ for some $n_i \in N$ for each $i = 1, ..., k$. But since $gn_ig^{-1} \in N_j \subseteq H$ for some $j = 1, ..., k$, we find that $ghg^{-1} \in H$ and thus that $H \lhd G$. By the minimality of M, this implies that $N_1...N_k = H = M$. Moreover, $N_i \cap N_j = \{1\}$ for each $i \neq j$, otherwise N_i and N_j wouldn't be minimal normal in M. Thus $M \cong N_1 \times \cdots \times N_k$. If there existed some $T \lhd N_1$ that is proper and nontrivial, then $T \times \{1\} \times \cdots \times \{1\} \lhd M$, contradicting the minimality of N_1. This of course extends to each N_i, and thus the N_i are simple and isomorphic to one another.

Now let $N_1, \ldots, N_r$ be the minimal normal subgroups of G, and let $N = N_1 \cdots N_r$. As we've shown, $N_i \cong S_i^{n_i}$ for some simple group S_i. The S_i is nonabelian, otherwise $\mathrm{Sol}(G) \neq 1$. Thus $Z(N_i) = 1$ for each $i = 1, ..., r$. By minimality, we have that if $i \neq j$, then $N_i \cap N_j = 1$, and so N_i and N_j commute with one another. Thus N_i and $\prod_{j \neq i} N_j$ commute with another. Since $N_i \cap \prod_{j \neq i} N_j \leq Z(N_i) = 1$, we see that $N \cong N_1 \times \cdots \times N_r$.

Since $N \lhd G$, the map taking $g \in G$ to the automorphism of N by g conjugation is a homomorphism from G into $\mathrm{Aut}(N)$ with kernel $C_G(N)$. Since the elements of $C_G(N)$ commute with the elements of N,

$$C_G(N) \cap N = Z(N) \cong Z(N_1) \times \cdots \times Z(N_r) = \{1\}.$$

Because $C_G(N)$ is normal in G as the kernel of a homomorphism, then $C_G(N) \neq \{1\}$ would imply that it contains some minimal normal subgroup M. But that would mean that $M \subseteq C_G(N) \cap N = \{1\}$. By contradiction, $C_G(N) = 1$ and thus G embeds into $\mathrm{Aut}(N)$ by the fundamental homomorphism theorem.

Moreover, since each N_i is normal in G, it is fixed by conjugation, meaning the image of G lies in

$$\mathrm{Aut}(N_1) \times \cdots \times \mathrm{Aut}(N_r) \leq \mathrm{Aut}(N_1 \times \cdots \times N_r).$$

□

Lemma 3.3 ([9], Main Theorem)**.** *Every nonabelian simple group, excluding* Alt_7*, has a character of degree divisible by* 4.

Theorem 3.4. *If G is a nonsolvable group satisfying the square-free character hypothesis, then $G/\mathrm{Sol}(G)$ is almost simple with nonabelian socle.*

Proof. By Mattarei's lemma, $S_1 \times \cdots \times S_k \lhd G/\mathrm{Sol}(G) \leq \mathrm{Aut}(S_1 \times \cdots \times S_k)$ where each S_i is nonabelian simple, not necessarily distinct. Of course, each $|S_i|$ is even by Feit-Thompson. Moreover, by Thompson's Theorem, there must exist some $\alpha \in \mathrm{Irr}(S) - \mathrm{Lin}(S)$ such that $\alpha(1)$ is odd for every nonabelian simple S. For the sake of contradiction, assume that $k > 1$ and that some $S_i \ncong \mathrm{Alt}_7$. Without loss of generality, assume $S_1 \ncong \mathrm{Alt}_7$ and by Lemma 3.3, choose $\lambda \in \mathrm{Irr}(S_1)$ to be a character of degree divisible by 4. Furthermore, let $\alpha \in \mathrm{Irr}(S_2)$ be a character of odd degree. It follows that $\chi = \lambda \times \alpha \times 1 \times \cdots \times 1 \in \mathrm{Irr}(S_1 \times \cdots \times S_k)$ by Theorem 4.21 of [5]. Since $4 \mid 7!/2 = |\mathrm{Alt}_7|$, we see that $16 \mid |S_1 \times \cdots \times S_k|$ $4 \mid \chi(1)$ and $\chi^c(1) = \frac{|S_1 \times \cdots \times S_k : \ker \chi|}{\chi(1)} = \frac{|S_1 \times \cdots \times S_k|}{\chi(1)}$ is also divisble by 4, contradicting the hypothesis. In the case that every $S_i \cong \mathrm{Alt}_7$, then $\chi = \alpha \times \alpha \times 1 \times \cdots \times 1 \in \mathrm{Irr}(S_1 \times \cdots \times S_k)$ is such that $4 \mid \chi(1)$ and is not divisible by 8. Therefore $4 \mid \chi^c(1)$ and $S_1 \times \cdots \times S_k$ fails the hypothesis. Moreover, by Corollary 3.1, $G/\mathrm{Sol}(G)$ must also fail the square-free character hypothesis. It follows that $k = 1$ and therefore that$G/\mathrm{Sol}(G)$ is almost simple with nonabelian socle. □

From here, it suffices for us to determine which nonabelian simple groups satisfy the hypothesis and then parsing through the possible almost simple groups. We first tackle the alternating groups.

Theorem 3.5 ([2], Theorem 1)**.** *If $\lambda = (\lambda_1, ..., \lambda_k)$ is a finite sequence of positive integers s.t. $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$, then for $n = \lambda_1 + ... + \lambda_k$, there exists an irreducible character χ_λ of Sym_n with degree $\frac{n!}{H(\lambda)}$. Furthermore, if $\lambda \neq \overline{\lambda}$, then χ_λ restricted to Alt_n is again irreducible and of the same degree. Here $H(\lambda)$ denotes the hook-length product and $\overline{\lambda}$ denotes the conjugate partition of λ.*

Lemma 3.6. *If $n \geq 8$, then $\mathrm{Irr}(\mathrm{Alt_n})$ contains characters of degree*

$$\frac{n(n-2)(n-4)}{3} \qquad \text{and} \qquad \frac{(n-1)(n-2)(n-3)}{6}$$

that extend to Sym_n.

Proof. Let $n \geq 8$ be arbitrary. If we let $\lambda = (n-3, 2, 1)$, then λ is not self-conjugate. By Theorem 3.1, we find that $\chi_\lambda(1) = \frac{n(n-2)(n-4)}{3}$. If we alternatively let $\lambda = (n-3, 1, 1, 1)$, then we similarly get a character corresponding to λ with $\chi_\lambda(1) = \frac{(n-1)(n-2)(n-3)}{6}$. □

It's not hard to see that for even $n \geq 8$, $\chi_\lambda(1) = \frac{n(n-2)(n-4)}{3}$ is divisible by 4 and has codegree also divisible by 4. The same could be said for odd $n \geq 8$ and the other character degree in Lemma 3.6. We now deal with the Lie type cases. As we'll see, the following theorem and corollary will be very important for our proof.

Theorem 3.7 ([24], Theorem A)**.** *Let $S = \mathrm{PSL}(2, q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leq H \leq A$. A is generated by S, a diagonal automorphism δ of order 2 if p is odd and 1 otherwise, and a field automorphism φ of order f. Set $G = \mathrm{PGL}(2, q)$ if $\delta \in H$ and $G = S$ if $\delta \notin H$, and let $|H : G| = d = 2^a m$, m odd. If p is odd, let $\varepsilon = (-1)^{(q-1)/2}$. The set of irreducible character degrees of H is*

$$\mathrm{c.d.}(H) = \{1, q, (q+\varepsilon)/2\} \cup \{(q-1)2^a \ell : \ell \mid m\} \cup \{(q+1)j : j \mid d\},$$

with the exceptions:

- *(1) $(q+\varepsilon)/2$ is not a degree if $p = 2$ or if p is odd with $H \not\leq S\langle\varphi\rangle$.*
- *(2) $\ell \neq 1$ if $p = 3$, f is odd, and $H = S\langle\varphi\rangle$.*
- *(3) $j \neq 1$ if $p = 3$, f is odd, and $H = A$.*
- *(4) $j \neq 1$ if $p = 2, 3,$ or 5, f is odd, and $H = S\langle\varphi\rangle$.*
- *(5) $j \neq 2$ if $p = 2$ or 3, f is even and $4 \nmid f$, and $H = S\langle\varphi\rangle$ or $H = S\langle\delta\varphi\rangle$.*

An immediate consequence of Theorem 3.7 is the following corollary.

Corollary 3.8.

$$\mathrm{c.d.}(\mathrm{PSL}(2, q)) \subseteq \left\{1, q-1, q, q+1, \frac{q+(-1)^{(q-1)/2}}{2}\right\}$$

H	$\chi(1)$	H	$\chi(1)$	H	$\chi(1)$
M_{11}	$2^2 \cdot 11$	He	$2^3 \cdot 5 \cdot 17$	Fi_{23}	$2^2 \cdot 3 \cdot 13 \cdot 23$
M_{12}	2^4	Ru	$2^2 \cdot 3^2 \cdot 7 \cdot 13$	Co_1	$2^2 \cdot 3 \cdot 23$
M_{22}	$2^3 \cdot 5 \cdot 7$	Suz	$2^2 \cdot 7 \cdot 13$	J_4	$2^3 \cdot 3^2 \cdot 23 \cdot 29 \cdot 37$
J_2	$2^2 \cdot 3^2$	ON	$2^6 \cdot 3^2 \cdot 19$	Fi_{22}	$2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 13$
M_{23}	$2^3 \cdot 11 \cdot 23$	Co_3	$2^7 \cdot 7$	Fi_{24}	$2^3 \cdot 7^2 \cdot 11 \cdot 17 \cdot 23 \cdot 29$
HS	$2^7 \cdot 7$	Co_2	$2^3 \cdot 11 \cdot 23$	B	$3^2 \cdot 5 \cdot 23 \cdot 31$
J_3	$2^2 \cdot 3^4$	HN	$2^3 \cdot 5 \cdot 19$	M	$2^2 \cdot 31 \cdot 41 \cdot 59 \cdot 71$
M_{24}	$2^2 \cdot 3^2 \cdot 7$	Ly	$2^4 \cdot 5 \cdot 31$		
McL	$2^2 \cdot 3^2 \cdot 7$	Th	$2^3 \cdot 31$		

TABLE 1. Characters of sporadic groups such that $\gcd(\chi(1), \chi^c(1))$ is not square-free.

Lemma 3.9. *If $S \lhd H \leq \mathrm{Aut}(S)$ with S being nonabelian simple and H satisfies the square-free character hypothesis, then S is one of the following groups:*

- Alt_n *for* $5 \leq n \leq 7$*;*
- J_1*;*
- ${}^2B_2(8)$*;*
- $\mathrm{PSL}(2, q)$ *for any prime power* q*.*

Proof. By checking the character tables of Alt_5, Alt_6, and Alt_7, we see that they all satisfy the square-free character hypothesis. As we've already shown, Lemma 3.6 implies that Alt_n fails the hypothesis for $n \geq 8$.
Moreover, we see that all sporadic groups not isomorphic J_1 have a counterexample character according to Table 1. Since $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, there cannot possibly exist a counterexample character.
Finally, for the simple Lie type groups, we refer to Tables 2 and 3 which describe the character degrees of every simple Lie type group not isomorphic to $\mathrm{PSL}(2, q)$ or ${}^2B_2(8)$. Checking the character table of ${}^2B_2(8)$, we can verify that it satisfies the hypothesis. And lastly, by Corollary 3.8, all groups $\mathrm{PSL}(2, q)$ also satisfy the hypothesis. □

The tables referenced in the proof of Lemma 3.9 were originally curated in [23]. The existence of these listed characters are gathered from various atlases and pieces of literature on the irreducible characters of simple Lie type groups [1][3][12][13][24]. Often times, we had to resort to using a computer algebra system to deal with various simple Lie type cases of small order and/or dimension [4].

From this point onward, we only need to check each of the almost simple groups with socle among the list described in Lemma 3.9.

H	$\chi(1)$	Factor of gcd
$A_2(3)$	$2^2 \cdot 3$	4
$A_2(4)$	$2^2 \cdot 5$	4
$A_2(q),\ q > 4,\ q \equiv 0 \bmod 4$	$q(q^2+q+1)$	4
$A_2(q),\ q > 4,\ q \equiv 1 \bmod 4$	$(q-1)(q^2+q+1)$	4
$A_2(q),\ q > 4,\ q \equiv 3 \bmod 4$	$(q+1)(q^2+q+1)$	4
$A_3(q)$	$q^3(q^2+q+1)$	q^2
$A_4(q)$	$q^6(q+1)(q^2+1)$	q^2
$A_n(q),\ n > 4$	$\dfrac{q^3(q^{n-1}-1)(q^n-1)}{(q-1)(q^2-1)}$	q^2
$B_2(q),\ q$ odd	$(q-1)(q^2+1)$	4
$B_2(q),\ q$ even, $q > 2$	$q(q+1)(q^2+1)$	4
$B_n(q),\ n > 2$	$\dfrac{(q^{n-2}-1)(q^{n-1}-1)(q^{n-1}+1)(q^n+1)}{2(q^2-1)^2}$	q^2
$C_n(q),\ n > 2$	$\dfrac{q^4(q^{2(n-2)}-1)(q^{2n}-1)}{2(q^2-1)^2}$	q^2
$D_4(2)$	$2^2 \cdot 7$	4
$D_4(3)$	$2^2 \cdot 5 \cdot 13$	4
$D_4(q),\ q > 3$	$\dfrac{1}{2}q^3(q+1)^4(q^2-q+1)$	q^2
$D_5(2)$	$2^2 \cdot 5 \cdot 17$	4
$D_5(q),\ q > 2$	$q^2(q^2+1)(q^4+1)$	q^2
$D_n(q),\ n > 5$	$\dfrac{q^6(q^{n-4}+1)(q^{2(n-3)}-1)(q^{2(n-1)}-1)(q^n-1)}{(q^2-1)^2(q^4-1)}$	q^2

TABLE 2. Characters of Lie type groups such that $\gcd(\chi(1), \chi^c(1))$ is not square-free.

It's worth mentioning that Tables 2 and 3 use Dynkin type notation when listing the groups. The projective special linear groups are denoted $A_n(q) = \mathrm{PSL}(n+1, q)$, the odd-dimension projective special orthogonal groups are denoted $B_n(q) = \mathrm{O}(2n+1, q)$, the projective symplectic groups are denoted $C_n(q) = \mathrm{PSp(2n, q)}$, and the even-dimension projective special orthogonal groups are denoted $D_n(q) = \mathrm{O}^+(2n, q)$.

H	$\chi(1)$	Factor of gcd
$E_6(q)$	$q^4\phi_2^3\phi_4^2\phi_6^2\phi_8\phi_{12}$	q^2
$E_7(q)$	$\frac{1}{2}q^3\phi_1^4\phi_3^2\phi_5\phi_7\phi_9\phi_{14}$	q^2
$E_8(q)$	$\frac{1}{2}q^4\phi_1^4\phi_3^2\phi_4^2\phi_5^2\phi_7\phi_9\phi_{10}\phi_{12}\phi_{15}\phi_{20}\phi_{30}$	q^2
$F_4(q)$	$q^2\phi_3^2\phi_6^2\phi_{12}$	q^2
$G_2(3)$	$2^6\cdot 3^6\cdot 7\cdot 13$	4
$G_2(4)$	$2^{12}\cdot 3^3\cdot 5^2\cdot 7\cdot 13$	4
$G_2(q),\ q>4,\ q\equiv 1 \bmod 2$	$q^2(q^2-q+1)(q^2+q+1)$	q^2
$G_2(q),\ q>4,\ q\equiv 2 \bmod 6$	$q^3(q-1)(q^2+q+1)$	q^2
$G_2(q),\ q>4,\ q\equiv 4 \bmod 6$	$q^3(q+1)(q^2-q+1)$	q^2
${}^2A_2(q),\ q>2,\ q\equiv 0 \bmod 4$	$q(q^2-q+1)$	4
${}^2A_2(q),\ q>2,\ q\equiv 1 \bmod 4$	$(q-1)(q^2-q+1)$	4
${}^2A_2(q),\ q>2,\ q\equiv 3 \bmod 4$	$(q+1)(q^2-q+1)$	4
${}^2A_3(q)$	$q^2(q^2+1)$	q^2
${}^2A_n(q),\ n>3$	$\dfrac{q^3(q^{n-1}-(-1)^{n-1})(q^n-(-1)^n)}{(q+1)(q^2-1)}$	q^2
${}^2B_2(q),\ q=2^{2n+1},\ n>1$	$(q-1)(q/2)^{1/2}$	4
${}^2D_4(q)$	$\frac{1}{2}q^3(q+1)^4(q^2-q+1)$	q^2
${}^2D_n(q),\ n>5$	$\dfrac{q^6(q^{n-4}+1)(q^{2(n-3)}-1)(q^{2(n-1)}-1)(q^n-1)}{(q^2-1)^2(q^4-1)}$	q^2
${}^3D_4(q^3)$	$q^7\phi_{12}$	q^2
${}^2E_6(q^2)$	$q^6\phi_3^2\phi_6^3\phi_{12}\phi_{18}$	q^2
${}^2F_4(2)'$	$2^2\cdot 3\cdot 5^2$	4
${}^2F_4(q),\ q=2^{2n+1},\ n>0$	$q\phi_6\phi_{12}$	4
${}^2G_2(27)$	$3^3\cdot 19\cdot 37$	9
${}^2G_2(q),\ q=3^{2n+1},\ n>1$	$(q/3)^{1/2}\phi_1\phi_2\phi_4$	9

TABLE 3. Character degrees for groups of Lie type cont. (For brevity in some cases, ϕ_n is the nth cyclotomic polynomial applied to q.)

For the finitely many cases that are not among the projective special linear groups, we can use a computer algebra system (GAP) to determine which almost simple groups satisfy the hypothesis. What remains are the infinitely many $\mathrm{PSL}(2,q)$. Using Theorem 3.7, we can derive the following lemma.

Lemma 3.10 ([23], Lemma 2.8). *Let H be a group such that $\mathrm{PSL}(2,q) \lhd H \leq \mathrm{Aut}(\mathrm{PSL}(2,q))$ where $q = p^f$ for some prime p. We have that $\mathrm{Aut}(\mathrm{PSL}(2,q))$ is generated by $\mathrm{PSL}(2,q)$, a diagonal automorphism δ, and a field automorphism φ of order f. Let $d = |H : \mathrm{PGL}(2,q)|$ if $\delta \in H$ and $d = |H : \mathrm{PSL}(2,q)|$ otherwise. H satisfies the square-free character hypothesis if and only if*

(1) If $p \neq 2$, then $p^2 \nmid d$;
(2) For any odd prime $r \neq p$, either $r^2 \nmid d$ or $r^4 \nmid d(q^2-1)$;
(3) $4 \nmid d$ in all cases, and in particular $2 \nmid d$ whenever $q \equiv 1 \pmod 8$, $q \equiv 7 \pmod 8$, or q is odd and $\delta \in H$.

The proof of Lemma 3.10 is rather technical. For keeping this subsection brief, we refer the reader to the original paper. Combining all our results thus far, we arrive at the following solution to the problem concerning nonsolvable groups satisfying the square-free character hypothesis.

Theorem 3.11 ([23], Theorem A). *Let G be a finite non-solvable group which satisfies the square-free character hypothesis. Then $G/\mathrm{Sol}(G)$ is isomorphic to one of the following almost simple groups.*

- J_1
- Alt_n *for* $n = 5, 6, 7$
- Sym_n *for* $n = 5, 6$
- M_{10} *or* $\mathrm{PGL}_2(9)$
- ${}^2B_2(8)$ *or* $\mathrm{Aut}({}^2B_2(8))$
- *All H such that* $\mathrm{PSL}(2,9) \lhd H < \mathrm{Aut}(\mathrm{PSL}(2,9))$
- *A group H where $\mathrm{PSL}_2(q) \lhd H \leq \mathrm{Aut}(\mathrm{PSL}_2(q))$ for $q = p^f \neq 9$ such that $d = |H : \mathrm{PGL}(2,q)|$ if the diagonal automorphism δ is in H and $d = |H : \mathrm{PSL}(2,q)|$ otherwise, which satisfies the following criterion*
  (1) If $p \neq 2$, then $p^2 \nmid d$;
  (2) For any odd prime $r \neq p$, either $r^2 \nmid d$ or $r^4 \nmid d(q^2-1)$;
  (3) $4 \nmid d$ in all cases, and in particular $2 \nmid d$ whenever $q \equiv 1 \pmod 8$, $q \equiv 7 \pmod 8$, or q is odd and $\delta \in H$.

Proof. By Theorem 3.4 and Lemma 3.9, we see that $G/\mathrm{Sol}(G)$ is almost simple with socle S among the list J_1, Alt_5, Alt_6, Alt_7, ${}^2B_2(8)$, and $\mathrm{PSL}(2,q)$ for q a prime power. If $S \cong \mathrm{PSL}(2,q)$ for $q \neq 9$, then we may directly apply Lemma 3.10. This covers all but finitely many cases for our choice of S. So we check the remaining cases by brute force using a computer algebra system (GAP). The resulting almost simple groups are listed in the statement of the theorem. $\square$

3.2. **Structural Results in the Solvable Case.** Bounding the Fitting length of a solvable group satisfying the square-free character hypothesis proves to be a remarkably difficult task. We instead take on a weaker open problem, which is to describe the nilpotent groups which satisfy the hypothesis. We also use the characterization of the nilpotent case to show that the derived length of a solvable case G has linear bound in terms of the Fitting length. In order to acquire these results, we'll need the following theorems.

Theorem 3.12 ([5], Theorem 12.11)**.** *A nonabelian group G has $\mathrm{cd}(G) = \{1, p\}$ for some prime p if and only if one of the following two conditions hold:*

- *There exists some abelian $A \lhd G$ such that $|G : A| = p$.*
- *$|G : Z(G)| = p^3$.*

Lemma 3.13 ([5], Corollary 12.9)**.** *Let $\mathrm{c.d.}(G) = \{1, p\}$ for some prime p. Then there exists some $A \lhd G$ such that $|G : A| = p$ or p^2.*

The key idea behind this next proof is that $\chi(1)$ must divide $\chi^c(1)$ for each $\chi \in \mathrm{Irr}(G)$ by the Gagola-Lewis theorem, thus $\gcd(\chi(1), \chi^c(1)) = \chi(1)$. It follows that for a nilpotent group, the square-free character hypothesis is equivalent to the fact that every $d \in \mathrm{c.d.}(G)$ is square-free. Fortunately, character theorists have already studied groups of this form, as is shown in Theorem 13.12 and Lemma 3.13.

Theorem 3.14. *A nilpotent group G satisfies the square-free character hypothesis if and only if every Sylow-p subgroup G_p is either*

- *abelian,*
- *contains some abelian $A \lhd G_p$ with $|G : A| = p$,*
- *or is such that $|G : Z(G)| = p^3$ and contains some abelian $A \lhd G_p$ with $|G : A| = p^2$.*

Proof. Suppose G is nilpotent and satisfies the square free-character hypothesis. As we've described, the Gagola-Lewis theorem guarantees that $\gcd(\chi(1), \chi^c(1)) = \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$. Thus, all elements of $\mathrm{c.d.}(G)$ must be square-free. Let G_p denote the unique Sylow-p subgroup of G (since G is nilpotent). Since $G_p \lhd G$, we notice that $\mathrm{c.d.}(G_p)$ must also satisfy the hypothesis, and thus $\mathrm{c.d.}(G_p) \subseteq \{1, p\}$. If $\mathrm{c.d.}(G_p) = \{1\}$, then G_p is abelian by Corollary 1.34. Otherwise, Theorem 3.12 informs us that either there is some abelian $A \lhd G_p$ such that $|G_p : A| = p$ or $|G_p : Z(G_p)| = p^3$. In the case that $|G : A| \neq p$ for all normal abelian A in G_p, then there must exist some $A \lhd G_p$ with $|G : A| = p^2$ by Lemma 3.13.
Conversely, $G = \prod_p G_p$ and thus every $\chi \in \mathrm{Irr}(G)$ is of the form $\chi = \prod_p \psi_p$ where $\psi_p \in \mathrm{Irr}(G_p)$ for each $p \in \pi(G)$. Since the character degrees of ψ_p are coprime to one another and square-free, they must either be prime or 1. Thus, $\chi(1)$ must also be square-free. Since $\gcd(\chi(1), \chi^c(1))$ divides $\chi(1)$, the condition of the square-free character hypothesis follows. □

Corollary 3.15. *If G is a solvable group satisfying the square-free character hypothesis, then $d(G) \leq 2h(G)$.*

Proof. Let $\{1\} = F_0 \lhd F_1 \lhd ... \lhd F_n = G$ be the Fitting series of G. Of course, each factor F_{i+1}/F_i is nilpotent. Since $d(G) = d(G/N) + d(N)$ for any $N \lhd G$, we may instead write $d(G) = \sum_{i=0}^{n-1} d(F_{i+1}/F_i)$, where $n = h(G)$. Evidently, our corollary follows so long as $d(F_{i+1}/F_i) \leq 2$ for each $i = 0, ..., n-1$. More generally, we prove that every nilpotent group satisfying the hypothesis has derived length ≤ 2.
Suppose H is such a nilpotent group and let H_p be the Sylow-p subgroup of H for each $p \in \pi(H)$. By Theorem 3.14, it follows that there exists some $A_p \lhd H_p$ with $|H_p : A_p|$ that divides p^2. But groups of prime and prime squared order are always abelian, hence $d(H_p) \leq 2$ for every $p \in \pi(G)$. Moreover, $H' = \left[\prod_{p\in\pi(G)} H_p, \prod_{p\in\pi(G)} H_p\right] = \prod_{p\in\pi(G)}[H_p, H_p] = \prod_{p\in\pi(G)} H_p'$. Since $d(H_p) \leq 2$, each H_p' must be abelian, and thus so is H'. As such $d(H) \leq 2$ and our result follows. □

If there exists a universal bound on the Fitting length of solvable groups satisfying the square-free character hypothesis, Corollary 3.15 tells us that a universal bound on the derived length must also exist.

4. A Potential Generalization of Gagola-Lewis

This section is dedicated to a first approach at Problem 5.2 in Guohua Qian's *Character codegrees in finite groups* [14]. The reader should notice how an affirmative answer to this problem is a generalization of the classical Gagola-Lewis theorem.

Problem: Suppose G is a group with $\pi(\chi(1)) \subseteq \pi(\chi^c(1))$ for every $\chi \in \mathrm{Irr}(G)$. We'll call these groups $\mathcal{I}$-groups. Must an $\mathcal{I}$-group be solvable? If so, what's a description of the $\mathcal{I}$-groups?

Remark 4.1. It's important to mention that an affirmative answer to this problem doesn't imply that it's an equivalent condition to solvability. As a matter of fact, the group $G = C_7 \rtimes C_3$ is a counterexample of order 21. Filling out the character table of G, we see that for $s = -\frac{\sqrt{7}+1}{2}$

$\mathrm{Cl}_G(g)$	1	x_1	x_2	x_3	x_4
χ_1	1	1	1	1	1
χ_2	1	ζ_3	ζ_3^2	1	1
χ_3	1	ζ_3^2	ζ_3	1	1
χ_4	3	0	0	s	$-s-1$
χ_5	3	0	0	$-s-1$	s

Both χ_4 and χ_5 are characters such that 3 divides the degree but not the codegree.

Remark 4.2. Another observation to note is that it suffices to check that $\exists \chi \in \mathrm{Irr}(H)$ such that $\pi(\chi(1)) \not\subseteq \pi(\chi^c(1))$ for $S^n \lhd H \leq \mathrm{Aut}(S^n)$ and any S that's nonabelian simple. This is due to a combination of Mattarei's lemma and Theorem 1.51. Notice how if $n = 1$, then we are simply working with almost simple groups. If G is a non solvable group with exactly one isomorphic copy of some nonabelian simple S among it's chief factors, we need only to check that the almost simple groups with socle S have some irreducible character χ such that $\pi(\chi(1)) \not\subseteq \pi(\chi^c(1))$ in order to confirm the same for G. In other words, the almost simple cases provide us with a great start to the problem.

Lemma 4.3. *If G is such that $S \trianglelefteq G \leq \mathrm{Aut}(S)$ for some simple group S that is either alternating or sporadic, then $\exists \chi \in \mathrm{Irr}(G)$ such that $\pi(\chi(1)) \not\subseteq \pi(\chi^c(1))$.*

Proof. For this lemma, we use the classification of finite simple groups to check each case. We start by supposing S is sporadic. By checking each case in [1], we see that there exists some $p \in \pi(S)$ such that $\gcd(|S|/p, p) = 1$. Moreover, $|\mathrm{Out}(S)| \leq 2$. In other words, $G = \mathrm{Aut}(S)$ or $G = S$. By Itô-Michler, it follows that there exists some $\chi \in \mathrm{Irr}(S)$ such that p divides $\chi(1)$. As such, p does not divide $\chi^c(1)$. Via Corollary 3.1, we see that there exists some $\psi \in \mathrm{Irr}(\mathrm{Aut}(S))$ such that $\chi(1)|\psi(1)$. Additionally, $|\mathrm{Aut}(S)| = 2|S| \implies \psi^c(1) = \frac{|\mathrm{Aut}(S)|}{\psi(1)}$ which is not divisible by p.
Now suppose $S \cong \mathrm{Alt}_n$ with $n \geq 5$. By Bertrand's postulate, there exists some prime $2 < \frac{n}{2} < p \leq n$. It follows that $\gcd(|\mathrm{Alt}_n|/p, p) = 1$, and in fact $\gcd(|\mathrm{Sym}_n|/p, p) = 1$. For $n = 5$ and $n \geq 7$, it's well known that $\mathrm{Aut}(\mathrm{Alt}_n) = \mathrm{Sym}_n$. So again, $G \cong \mathrm{Alt}_n$ or $G \cong \mathrm{Sym}_n$. In either case, we resort to Itô-Michler to find a character $\chi \in \mathrm{Irr}(\mathrm{Alt}_n)$ such that p divides $\chi(1)$. In the case that $n = 6$, the almost simple groups with socle Alt_6 include Alt_6, Sym_6, M_{10}, and $\mathrm{PGL}(2, 9)$. Once again, there exists a $\chi \in \mathrm{Irr}(\mathrm{Alt}_6)$ with p dividing $\chi(1)$. Selecting any $\psi \in \mathrm{Irr}(G|\chi)$ suffices since $|G|$ incurs a factor of either 2 or 4. □

Lemma 4.4 ([25])**.** *If $G = L(q)$ denotes a simple Lie type group over a field of order $q = p^f$, then there exists some $\mathrm{St}_G \in \mathrm{Irr}(G)$ such that $\mathrm{St}_G(1) = |G|_p$.*

Lemmas 4.3 and 4.4 allow us to state the following theorem and corollary.

Theorem 4.5. *A nonabelian simple group is not an $\mathcal{I}$-group.*

Corollary 4.6. *Every $\mathcal{I}$-group has a nontrivial linear character.*

Proof. Suppose that $\mathrm{Lin}(G) = \{1\}$. If $G' < G$, then we'd have that every $\chi \in \mathrm{Irr}(G/G')$ determines an irreducible linear character of G. By contradiction, we must have that $G' = G$ and thus that every maximal normal subgroup $M \lhd G$ is such that G/M is nonabelian simple. By Theorem 4.5, we see that G/M is not an $\mathcal{I}$-group. But the property of being an $\mathcal{I}$-group is inherited by quotients, so G is also not an $\mathcal{I}$-group. By contradiction, $\mathrm{Lin}(G)$ is nontrivial. □

The character described in Lemma 4.4 is called the **Steinberg** character of $L(p^f)$. It's particularly useful to us since it often yields a character in the almost simple groups with socle $L(p^f)$ that has p-part equal to the p-part of the group's order. However, we must circumnavigate the cases when $p \mid |\mathrm{Out}(L(p^f))|$. Unlike the alternating and sporadic groups, Lie type simple groups can have large outer automorphism groups. This is particularly an issue when $p \mid f$, since $|\mathrm{Out}(L(p^f))|$ is always divisible by f. Fortunately, the Suzuki groups and Ree groups over characteristic 2 give us a break.

Lemma 4.7. *If H is an almost simple group with socle isomorphic to any group in the following list, then H cannot be an $\mathcal{I}$-group.*

- $\mathrm{PSL}(2, q)$
- ${}^2B_2(2^f)$ *with* $f = 2n + 1$
- ${}^2F_4(2^f)$ *with* $f = 2n + 1$.

Proof. The $\mathrm{PSL}(2, q)$ case follows from Theorem 3.7 by choosing the character of degree $(q + 1)d$, where $d = |H : \mathrm{PGL}(2, q)|$ when the diagonal automorphism is in H and $d = |H : \mathrm{PSL}(2, q)|$ otherwise.
Moreover, the ${}^2B_2(2^f)$ and ${}^2F_4(2^f)$ cases follow since the outer automorphism groups of each are equal to f, which is always odd. Thus our usual strategy of applying Corollary 3.1 functions without issue. □

By Lemmas 4.3 and 4.7, as well as Mattarei's lemma, the following theorem follows almost immediately.

Theorem 4.8. *If G is a nonsolvable group with precisely 1 nonabelian chief factor S, then G is not an $\mathcal{I}$-group whenever S is*

- *Sporadic,*
- Alt_n*,*
- $\mathrm{PSL}(2, q)$*,*
- ${}^2B_2(2^f)$ *with* $f = 2n + 1$*,*
- *or* ${}^2F_4(2^f)$ *with* $f = 2n + 1$*.*

Though we state Theorem 4.8, it's important that the reader doesn't think that the sole purpose of checking whether all almost groups are not $\mathcal{I}$-groups is to prove that all nonsolvable groups with exactly one nonabelian chief factor are not $\mathcal{I}$-groups. As we've hinted in Remark 4.2, the statement that for any nonabelian simple S, each H such that $S^n \lhd H \leq \mathrm{Aut}(S^n)$ is not an $\mathcal{I}$-group is equivalent to an affirmative answer to the open problem. Since $\mathrm{Aut}(S^n) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}_n$, the problem is very closely related to studying the almost simple groups in particular.

Conjecture 4.9. *Every almost simple group with nonabelian socle is not an $\mathcal{I}$-group.*

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