Regular Maps

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Spring 2025

Abstract

This exposition is dedicated to summarizing findings in the study of regular map that aims to generate both a basic understanding of and an interest in the subject. We cover various perspectives to defining maps and regularity condition for regular maps. We summarize findings by many authors in expanding the collection of regular maps and the properties of these maps.

1 Introduction

In mathematics, we have seen the abuse of terms by assigning one English word to too many different objects. We often rely on the context of the field in which the word present itself. In our case, the word is "map." We want to first make clear that we are not referring to the assignment of elements from one set to another set that essentially forms a set of ordered pairs. The term "regular map" in many instances are used to describe a class of such maps. They are not the object of our discussion. Here, we take the path of first explaining our object of study before describing its history and applications. There are many ways in which the literature has defined a map, and thus we give a survey of competing definitions.

1.1 Maps

One simple definition of a map is by describing it as a 2-cell embedding of a connected graph into a closed surface. Here, we involve two other objects in the definition: a connected graph and a closed surface. We first define these two objects. A connected graph $G = (V, E)$ is composed of a set of vertices V (each vertex can be thought of as a point) and a set of edges E (each edge can be thought of as a pair of points). We further define the notion of a face being a circuit of vertices where they share an edge. In [1], Conder defines a face alternatively as a simply-connected component of the space obtained by removing the graph. Some authors prefer the inclusion of the edges between the vertices and separately the repetition of the first vertex at the end of the circuit.

Another choice one could make is whether to impose that these graphs must be simple though others may prefer double edges in their definition. Finally, we impose that the graph is connected in the sense that we can travel between any two vertices along some sequence of edges which we call the path between two vertices. Thus far, we have a connected graph which can be realize in some $\mathbb{R}^n$ space. However, we want to generalize this embedding into other surfaces, which are called manifolds. We give the following definition, slightly restricted than the more common definition.

Definition 1. *A subset M of $\mathbb{R}^n$ is a manifold if M is locally $\mathbb{R}^k$ for some $k \leq n$, i.e. for every point $p \in M$, we can find an open neighborhood $U \subseteq M$ at p and a continuous map $\varphi : U \to \mathbb{R}^k$.*

The restriction to subset of $\mathbb{R}^n$ allows us to abstract from the notions of Hausdorffness and second-countability which are essential conditions on a manifold but not directly related to our discussion.[1] We use the term surface to describe a 2-manifold. Further, we demand the surface to be compact. The orientatability of this map is given by the underlying surface we embed the graph on.

An alternative but highly related formulation is given by Wilson [13]. Wilson first defines the surface then partitioning the surface with a set of arcs. The partitions become faces, and the intersections of the arcs become vertices. We have so far used a constructive approach to the definition. We appeal to a different constructions of maps, as cell-complexes. In particular, we introduce maps as CW-complexes. We first define a cell.

Definition 2 (Cells)**.** *For any integer $k \geq 0$, a k-cell D is a topological space homeomorphic to the closed Euclidean ball of radius 1 about the origin in $\mathbb{R}^d$.*

Having described the cell, we describe a type of complex called the CW-complex. We will build these complexes recursively, starting with the CW_0-complex being the set of vertices. Then, we define a way to "glue the pieces" of these complexes together. Formally,

Definition 3 (CW-complex)**.** *Let X be a set (of points). The CW_0-complex $X_0 = X$, the set of* 1*-cells, i.e. points. Given a CW_{k-1}-complex X_{k-1}, a set of k-cells $\{C_{k,\alpha}\}_\alpha$, and the gluing maps $d_\alpha : \partial C_{k,\alpha} \to X_{k-1}$. Then the CW_k-complex is defined as $X_k = X_{k-1} \sqcup \{C_{k,\alpha}\}_\alpha$.*

For our purpose, we will only need up to CW_2-complex. We are interested in complexes whose 2-cells "cover" the surface, i.e. the map is a 2-cell decomposition of the surface. This definition ground our notions of vertices, edges, and faces to the 0-, 1-, and 2-cells in the CW_2-complex. It does not add much intuition to the understanding of maps, but we allow, through describing it as a complex, the machinery we work with complexes. While this definition ties the three objects as different dimensional cells, this needs not be the case.

We diverge from the constructive approach thus far and utilize the axiomatic approach. We further recognize that the theory of maps is interlinked with the theory of polyhedra. Every polyhedra naturally defines a map on its surface, but of course we can define a different map. The language of polyhedra can thus be adapted to discuss maps. We will tend to abuse the distinction between the two. In such case, we mainly refer to the natural choice of map of a polyhedron.

A polyhedra can be thought of as a set of polygons where every edge is glued to one other polygon. Indeed, this is the general definition for polyhedra at the early stage of its theory. Coxeter, Higgins, and Miller used this definition in [2]. The theory of polyhedra has since taken on new abstract definitions, and we will provide two different axiomatic definitions of an abstract polyhedra. These definitions are given by influential mathematicians within this field. Of course, the contributions of Coxeter cannot be understated.

The first definition is by Grünbaum in [6].

Definition 4 (Polyhedra – Grünbaum)**.** *In an abstract polyhedron $\mathcal{P} = (V, E, F)$, where $V = \{V_i\}$ is the vertex set, $E = \{E_j\}$ is the edge set, and $F = \{F_k\}$ is the face set.*

*(P1) Each edge is incident with precisely two distinct vertices and two distinct faces. Each of the two vertices is said to be **incident** (via the edge in question) with each of the two faces. Two vertices incident with an edge are said to be **adjacent**; also, two faces incident with an edge are said to be **adjacent**.*

[1] In fact, every compact metrizable space X of topological dimension m can be embedded in R^{2m+1}. ([11], pp. 309) Thus, the definition is sufficient to describe all manifolds we discuss which are compact.

(P2) For each edge, given a vertex and a face incident with it, there is precisely one other edge incident to the same vertex and face. This edge is said to be ***adjacent*** *to the starting edge.*

(P3f) For each face there is an integer k, such that the edges incident with the face, and the vertices incident with it via the edges, form a circuit in the sense that they can be labeled as $V_1E_1V_2E_2V_3E_3 \dots V_{k-1}E_{k-1}V_kE_kV_1$, where each edge E_i is incident with vertices V_i and $Vi+1$, and adjacent to edges E_{i-1} and E_{i+1}. All edges and all vertices of the circuit are distinct, all subscripts are taken mod k, and $k \geq 3$.

(P3v) For each vertex there is an integer j, such that the edges incident with the vertex, and the faces incident with it via the edges, form a circuit in the sense that they can be labeled as $F_1E_1F_2E_2F_3E_3 \dots F_{j-1}E_{j-1}F_jE_jF_1$, where each edge E_i is incident with faces F_i and F_{i+1}, and adjacent to edges E_{i-1} and E_{i+1}. All edges and all faces of the circuit are distinct, all subscripts are taken mod j, and $j \geq 3$.

Thus, each face corresponds to a simple circuit of length at least 3, and similarly for the circuits that correspond to the vertices; the latter circuits are known as ***vertex stars***.

(P4) If two edges are incident with the same two vertices [faces], then the four faces [vertices] incident with the two edges are all distinct.

(P5f) Each pair F, F^* of faces is connected, for some j, through a finite chain

$$F_1E_1F_2E_2F_3E_3 \dots F_{j-1}E_{j-1}F_j$$

of incident edges and faces, with $F1 = F$ and $F_j = F^*$.

(P5v) Each pair V, V^* of vertices is connected, for some j, through a finite chain

$$V_1E_1V_2E_2V_3E_3 ... V_{j-1}E_{j-1}V_j$$

of incident edges and vertices, with $V_1 = V$ and $V_j = V^*$.

This definition treats vertices, edges, and faces as three independent objects that abstracts from the associations to points, arc, and regions. The definition is also interesting in the symmetry between vertices and faces: the axioms apply equally to both of them. This is nice property as we later discuss dual maps.

The second axiomatic definition for a polyhedra relies on the concepts of faces and abstracts from the notions of edges and vertices. The definition is due to McMullen and Schulte in [10].

Definition 5 (Polyhedra – McMullen and Schulte)**.** *A polyhedron $\mathcal{P}$ is a partially ordered set $(F(P), \leq)$ that satisfies*

(P1) $\mathcal{P}$ contains a least face and a greatest face; they are denoted by F_{-1} and F_n, respectively.

(P2) Each flag of $\mathcal{P}$ has length $n+1$ (that is, contains exactly $n+2$ faces including F_{-1} and F_n).

(P3) $\mathcal{P}$ is strongly connected, in the sense that each section $G/F := \{H | H \in \mathcal{P}, F \leq H \leq G\}$ of $\mathcal{P}$ is connected.

(P4) For each $i = 0, 1, \dots, n-1$, if F and G are incident faces of $\mathcal{P}$, of ranks $i-1$ and $i+1$, respectively, then there are precisely two i-faces H of $\mathcal{P}$ such that $F < H < G$, where we define rank $F :=$ rank F/F_{-1} and an i-face is a face with rank i.

A flag according to [10] is a the maximal chain of $\mathcal{P}$, which is a totally ordered subset of $\mathcal{P}$. McMullen and Schulte also gave an alternative condition of (P3)

(P3') $\mathcal{P}$ is strongly flag-connected, in the sense that every section of $\mathcal{P}$ is flag-connected. A section is flag-connected if any two flags Φ, Ψ can be connected by a sequence of adjacent flags

$$\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$$

Two flags are adjacent if they differ by one face.

These definition brings groundings to study of abstract polyhedra. We want to draw a distinction between maps and polyhedra. While the study of polyhedra and the natural map is informative, there are of course different maps we can identified on the poylyhedra, by first thinking of it as a surface and then embed a connected graph on that surface. Figure 1 illustrates this point. The first map is the natural choice.

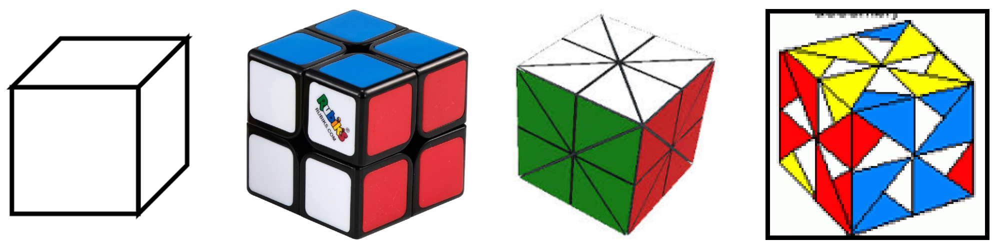

Figure 1: Different maps on a cube.

Remark 6. *While we can define a regular map without reference to a surface (as in the case with the CW-complex construction), we will often refer to the surface to guide our discussion.*

Finally, we further defines two other objects related to maps: darts and flags. Darts are simply a pair of vertex and edge. While flag has been defined previously in the context of the definition of abstract polyhedra in [10], we employ a more conventional definition of flag as below, which is informative enough at the sacrifice of specificity to different definitions. As flags will become important in defining regular maps, we record the following definition, as mentioned in [14].

Definition 7 (Flag)**.** *A flag in a map is a mutual incidence of a face, an edge and a vertex.*

It is easy to see that every dart consists of two flags, each defined on the two faces incident the dart, except possibly in the case where the edge lies only in one face. Those cases will not concern us.

Figure 2 illustrates a sets of flags. We will describe a flag as (face edge vertex) following Wilson [14]. Some examples of flags in Figure 2 are (A a 2), (B b 3), and (A b 3). Flags become important in the following section as regularity is defined with flags and being an incidence of the three objects (vertices, edges, and faces) means that a condition placed on flags is quite strong and induce nice properties.

1.2 Historical Context

It is worth to consider the history of the theory of polyhedra, a motivator for regular maps, before discussing what consitutes a regular map. Grünbaum [6] gives an account of the phenomenon that successive mathematicians continually prove that the most recent list of identified polyhedra to them exhausts all polyhedra, only to have new polyhedra found in a few decades. We will discuss here an adapted version of that discussion.

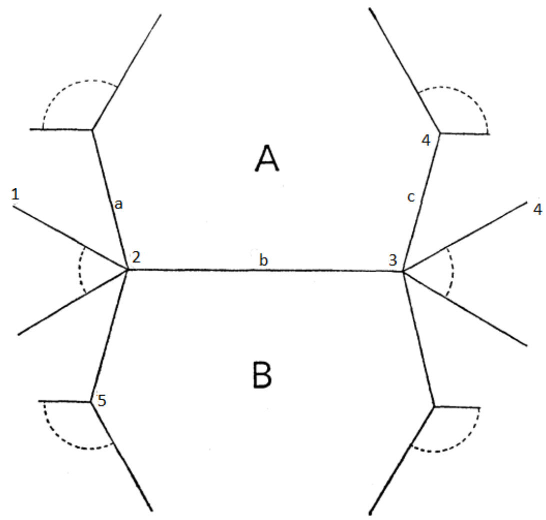

Figure 2: Flags in regular map, adapted from [14]

The theory of polyhedra can be traced back to the ancient Greek who studied the five Platonic solids, as mentioned in the *Elements* by Euclid. Figure 3 gives an illustrations of these solids. Other names include regular convex polyhedra and regular convex polytopes. For a long time, these are considered the complete list of all regular polyhedra.

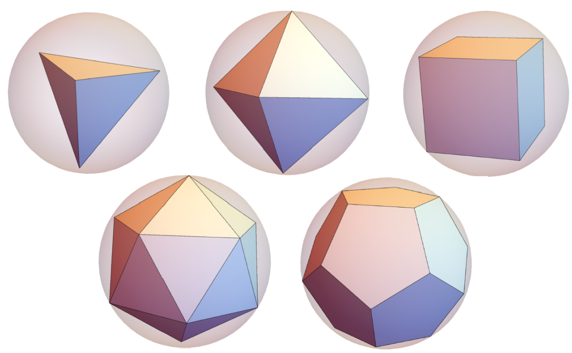

Figure 3: The five Platonic solids as studied by the ancient Greeks. From top row and left to right: Tetrahedron, Octahedron, Cube, Icosahedron, Dodecadedron.

Kepler and Poinsot thousands of years later found four more polyhedra, often known as Kepler-Poinsot star polyhedra. These polyhedra, as shown in Figure 4, allow for intersecting faces. Thus, there are vertices-like points which are not vertices of the polyhedra. It may be more advantageous to imagine them as four-dimensional polyhedra embedded in three-dimensional space, and thus the faces indeed do not intersect. This diverges from the understanding of polyhedra as the ancient Greeks did. As Grünbaum noted, Cauchy later showed that there are no other polyhedra, at least in the sense understood so far. Later Petrie and Coxeter introduced in [3] a new kind of polyhedra called regular skew polyhedra. Some examples are given in Figure 5.

We want to point to a slightly different map, tesselation of the plane. The following tessellations of the Euclidean plane with regular polygons and were noticed by Kepler as analogues to the polyhedra, as noted by Coxeter in [3], presented in Figure 6. It turns out that these tessellations under certain group actions forms a regular map on the torus. We defer this discussion to the end when more machinery has been defined.

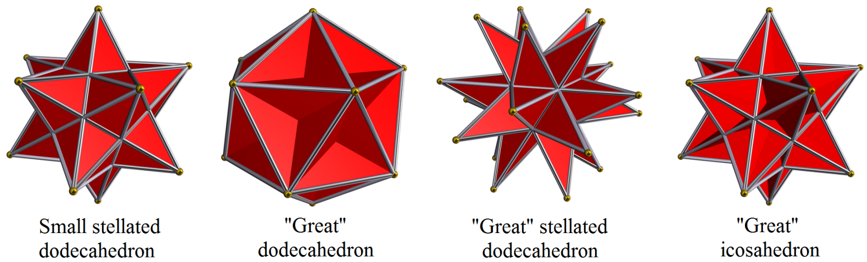

Figure 4: Kepler-Poinsot polyhedra.

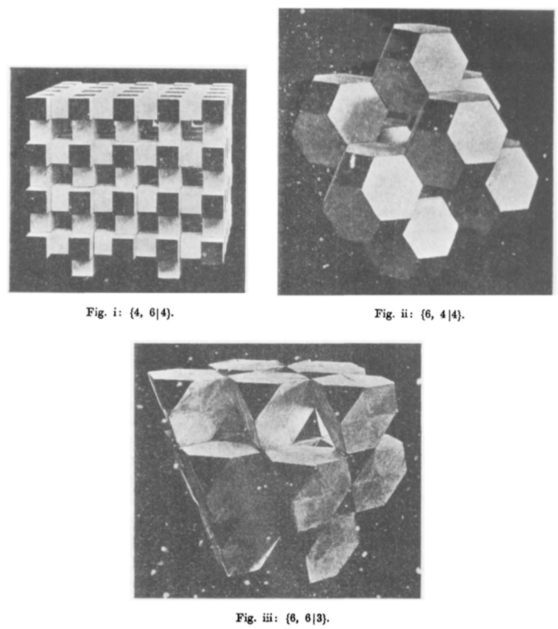

Figure 5: Regular skew polyhedra found by Petrie and Coxeter as introduced in [3].

1.3 Regular Maps

The condition of regularity is brought about to narrow our scope of study to "nice" objects. The specific form of regularity has changed throughout the years and differs from author to author. We will build up the theory from weaker forms of regularity to stronger forms.

Let us introduce an important relationship between the vertex set V, edge set E, and face set F. Before that, we define $D : V \to \mathbb{R}$ as the function mapping every vertex v to the number of edges surrounding v. We call this the degree of v. We also define $F_j = \{f \in F : f = j\text{-gon}\} \subseteq F$ to be the set of j-gons in the face set. Now, we can state the following proposition.

Proposition 8. *Let $D(v)$ be the degree of vertex $v \in V$ and $F_j \subseteq F$ be the collection of j-gon. Then*

$$\sum_{v \in V} D(v) = 2|E| = \sum_{j=3}^{|V|} j|F_j|$$

Proof. For every edge, there are two endpoints. Thus, the sum of the numbers of edges connected to each vertex double counts the edges, giving the first equality. Similarly, for every edge, each face is incident by two faces. $j|F_j|$ gives the total number of edges surrounding

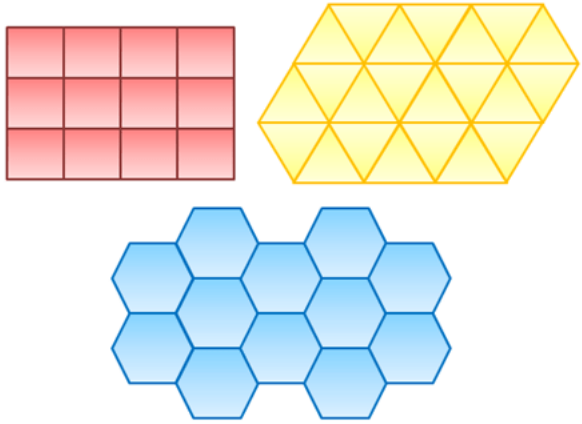

Figure 6: Tessellations of the Euclidean plane with regular polygons.

the j-gons. Thus, the last part of the equation describes the total number of edges but again double counts every edge. Hence, we have the second equality. $\square$

Proposition 8 gives a relationship between the three sets V, E, F. We now consider first form of regularity. We can demand some "nice" property where we could simplify this relationship. In particular, we want every face to have the same number of vertices/edges, i.e., a map of some q-gon. We can also ask that every vertex is surrounded by some p faces/edges. Some author refers to p as the valency of the vertex and q as the co-valency of the face, as used in [13]. The result is that we have the following.

$$p|V| = q|F| = 2|E| \tag{1}$$

This condition turns out to be quite a weak condition that permits certain maps that we might not be interested in. The reason is that they impose no condition on symmetries or length. Thus, a tesselation of slanted triangles would be considered a nice map, but they admit no rotational symmetry except for the identity map (and perhaps the 180^0 rotation). Wilson [13] describes these maps as uniform.

We want to relate to the discussion of regularity of polyhedra. Grünbaum [6] essentially defines three classes of polyhedra based on the symmetries they admit. Here we find our definition is based on symmetry. Grünbaum gives the following definitions.

Definition 9 (Symmetries on Polyhedra – Grünbaum)**.** *A symmetry of a (geometric) polyhedron is a pairing of an isometric mapping of the polyhedron onto itself with an automorphism of the underlying abstract polyhedron. The polyhedron is isogonal [isohedral, regular] if its vertices [faces, flags] form one orbit under its symmetries.*

Grünbaum defines uniform polyhedra as isogonal polyhedra with regular faces, similar to the understanding of Coxeter, Higgins, and Miller [2]. The faces are not required to have the same number of sides. Examples of uniform polyhedra are the thirteen Archimedean solids, which are shown in Figure 7. We note here the significance of [2] in discussing uniform polyhedra, reigniting the field of abstract polyhedra, and motivating many authors in their pursuit of this field, as noted by Grünbaum in [7].

Grünbaum further describes a class of polyhedra called noble polyhedra, which are polyhedra that are both isogonal and isohedral. These maps were studied by Hess and Brückner (see [6] for a longer discussion). Figure 8 gives some examples of non-regular noble polyhedra.

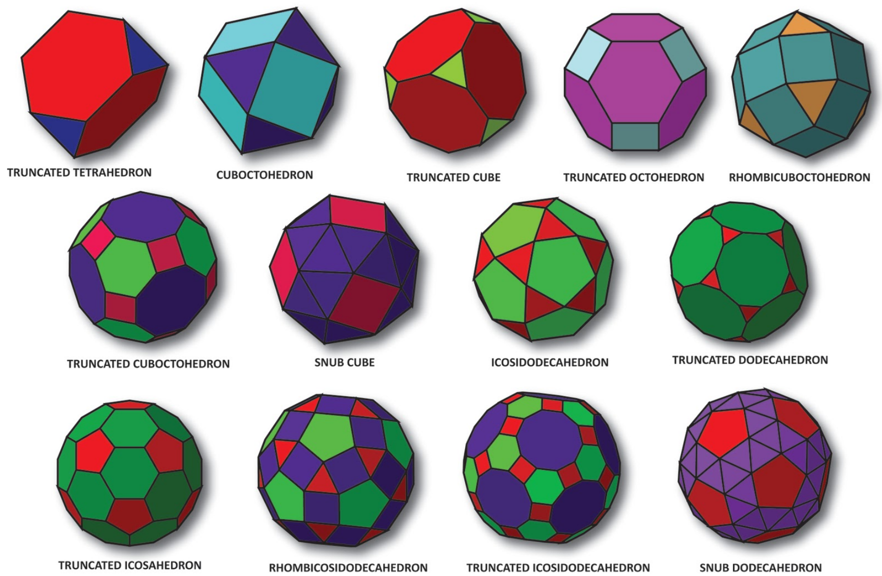

Figure 7: Archimedean solids reintroduced by Coxeter, Higgins, and Miller [2].

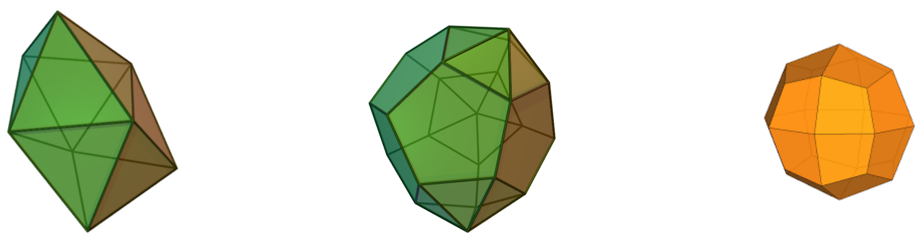

Figure 8: Polyhedra that are noble but not regular.

Finally, Grünbaum defines regular polyhedra to form an orbit under symmetries of the flags. In the language of regular maps, Wilson [13] and Conder and Dobcsányi [1] describes such maps as being flag-transitive. We offer the following definition.

Definition 10 (Regular Maps)**.** *A map is called regular provided that its group of symmetries acts transitively on its flags.*

We now recall a previously mentioned desired property of regular maps. We restate it as the following proposition and provide a proof.

Proposition 11. *Given regular map M. Let V, E, F be vertex, edge, and face set of M. Then Equation 1 is true, i.e. for some positive integers p, q,*

$$q|V| = 2|E| = p|F|$$

Proof. By contradiction, assumes that a regular map has two faces [vertices] surrounded by p, p' edges, where $p \neq p'$. Then the rotation R^p $[S^p]$ is the identity map around the first face [vertex] but this implies that it is not the identity map around the second face [vertex]. Hence, the symmetries do not act transitively on all flags. The contradiction implies that, in regular maps, faces [vertices] have equal number of surrounding edges. $\square$

So far, we have defined and represented regular maps using the numbers of vertices, edges, and faces. However, based on Proposition 11, we can get the numbers of vertices and faces by $p, q, |E|$ as follows

$$|F| = \frac{2|E|}{p}, \quad |V| = \frac{2|E|}{q} \tag{2}$$

Thus, we can refer to a regular map with either $(|V|, |E|, |F|)$ or with $(\{p, q\}, |E|)$. Often, the latter is more informative to the regular maps we are inspecting. We will introduce in later sections other ways to represent regular maps.

1.4 Automorphisms on Regular Maps

While our definition of a regular is generally accepted by the literature, authors often differ on the kinds of symmetries they would admit. We call these symmetries automorphisms, i.e. homomorphisms[2] from a map to itself. We now proceed to describe exactly what kinds of symmetries admitted by a regular map. The discussion is adapted from that of Wilson's in [14] and closely observe the flags in Figure 2.

We first describe several reflections. a symmetry α which interchanges the flags (A b 2) and (A b 3), another, β, which interchanges (A b 3) and (B b 3), and a third, X, which interchanges (A b 3) and (A c 3). Wilson [14] defines the symmetries on a regular maps to be these reflections. This implies that a regular map also acts transitively on its flags under the rotations: $R = \alpha X = X \circ \alpha$ (rotate one-step clockwise around the face A), $S = \beta X$ (rotate one-step clockwise around the vertex 3), and $\gamma = \alpha\beta$ (rotate 180^0 around the edge b). We summarize the relation between different symmetries.

$$\alpha^2 = \beta^2 = X^2 = R^p = S^q = \gamma^2 = 1 \tag{3}$$

We diverge from Wilson's discussion here to describe symmetries as presented in Conder and Dobcsányi to help with later discussion. Instead of reflections, we can begin with rotations R, S. Then $RS = \gamma$. Then we can define our first reflection, $a = \alpha$. The other reflections follow $b = aR = X$ and $c = bS = \beta$ (the latter equality by replacing S with its inverse). We summarize their relation below.

$$R^p = S^q = (RS)^2 = a^2 = b^2 = c^2 = 1 \tag{4}$$

We thus have two sets of automorphisms: reflections and rotations. Wilson in his definition requires a map to admit both symmetries to be considered regular. We further observes that because rotations are compositions of reflections, satisfying the latter would satisfy the former but not vice versa. Thus, we could impose a weaker condition than that of Wilson.

Indeed, as noted by Wilson, some authors prefer a less strict condition for regularity, where the only symmetries imposed are rotations. This is the definition adopted in Conder and Dobcsányi [1]. In particular, Conder and Dobcsányi refer to Wilson's regular maps as reflexible regular maps, while the regular maps admitting symmetries on rotations but not the reflections are called chiral regular maps. Figure 9 shows examples of these two kinds of regular maps. Under reflections, Figure 9(a) remains the same, while a reflection to Figure 9(b) on edge b and d (a verticle reflection of the tesselation) will create a new map.

As we presently see no benefit in demanding reflective symmetries and the fact that there are many chiral regular maps that are interesting, we adopt the definition of regular maps in the sense of Conder and Dobcsányi [1].

We can further look at a map's orientability[3]. Coxeter and Moser [4] showed that there

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[2] The requirement of homomorphism on serves to ensure the associativity of the actions.

[3] We employ an intuitive notion of orientability. If one can assign one orientation to every face of a map M, either clockwise or anticlockwise, then the map is orientable. Otherwise, the map is non-orientable. Generally, the map's orientability is the same as that of the surface. Examples of non-orientable surfaces are Mobiüs strip and Klein bottle.

(a) Reflexible

(b) Chiral

Figure 9: Toroidal Maps.

exists no non-orientable chiral map. Thus, we can subdivide regular maps, which is used by Conder and Dobcsányi [1], as

- Orientable and Reflexible,
- Orientable and Chiral, or
- Non-orientable

We return to our discussion symmetries. Let us define the set of all symmetries admitted by M as the automorphism group of M, denoted as $\operatorname{Aut} M = \langle \alpha, \beta, X \rangle$, generated by the reflections. Let $\operatorname{Aut}^+ M = \langle R, S \rangle \subseteq \operatorname{Aut} M$ be the rotation subgroup of M. There are precisely the orientation-preserving automorphisms. We present below the relationship between the two groups for each kind of regular map. The discussion is adapted from that in [1].

- If M is orientable and reflexible, then $\operatorname{Aut}^+ M$ is a normal subgroup of index 2 in $\operatorname{Aut} M$.
- If M is orientable and chiral, there is no reflection, thus $\operatorname{Aut} M = \operatorname{Aut}^+ M$.
- If M is non-orientable, every reflection is a rotation and thus $\operatorname{Aut}^+ M$ is a normal subgroup of index 1 in $\operatorname{Aut} M$.

2 Euler Characteristic

First introduced by Euler for convex polyhedra, but it can be defined more generally for any polyhedra. We introduce the simple form of the Euler characteristic.

Definition 12 (Euler characteristic – simple form)**.** *The Euler characteristic of a polyhedron* $X = (V, E, F)$ *is*

$$\chi(X) = |V| - |E| + |F|.$$

Recall our earlier discussion that we can refer to a regular map with an alternative formulation in terms of $p, q, |E|$. We can provide an alternative relation between those terms and the Euler characteristic. Given a regular map $X = (\{p, q\}, |E|)$, the Euler characteristic is

$$\chi(X) = \frac{2}{q}|E| - |E| + \frac{2}{p}|E|$$

Alternatively, we can write

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{q} = \frac{\chi(X)}{2|E|} \tag{5}$$

A result from this formulation is that given $\{p, q\}$ and the Euler characteristic of a regular map, we can identify a regular map as $(\{p, q\}, \chi)$ instead of $(|V|, |E|, |F|)$.

While the simple form definition is sufficient for our discussion, there is a generalization over cell complexes (of higher order than 2). We provide that definition here.

Definition 13 (Euler characteristic – general form)**.** *Given a finite chain complex $X = (C_k, \partial)_{k=0}^n$, the Euler characteristic is*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim C_k$$

In risking not treating the homology theory justice, we introduce the following proposition. The key point is that homology is homotopy invariant.

Proposition 14. *Given a chain complex (C_k, ∂). Let $Z_k = \{x \in C_k : \partial x = 0\} = \ker \partial_k$ and $B_k = \{\partial x : x \in C_{k+1}\} = \operatorname{Im} \partial_{k+1}$. Define the homology on the complex chain is $H_k = Z_k/B_n$. Then*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim H_k = \sum_{k=0}^{n} (-1)^k b_k$$

We call $b_1, \ldots, b_n$ the Betti numbers.

Proof. Because $H_k = Z_k/B_k$, $\dim H_k = \dim Z_k - \dim B_k$. By rank-nullity theorem, we have

$$\dim C_k = \dim \ker \partial_k + \dim \operatorname{Im} \partial_k = \dim Z_k + \dim B_{k-1} = \dim H_k + \dim B_k + \dim B_{k-1}$$

Hence, in our sum we have

$$\sum_{k=0}^{n} (-1)^k (\dim H_k + \dim B_k + \dim B_{k-1}) = \sum_{k=0}^{n} (-1)^k \dim H_k + \sum_{k=0}^{n} (-1)^k (\dim B_k + \dim B_{k-1})$$

The latter terms sum to $(-1)^n \dim B_n + \dim B_{-1} = 0$, so we obtain the desired equation. □

We further provide the following proposition relating genus g of a map M to its Euler characteristic $\chi(M)$, and the orientation of M.

Proposition 15. *For a map M with genus g, the Euler characteristic is*

$$\chi(M) = |V| - |E| + |F| = \begin{cases} 2 - 2g & \text{if } M \text{ is orientable} \\ 2 - g & \text{if } M \text{ is non-orientable.} \end{cases}$$

This proposition is important to show the inextricable link between genus and the Euler characteristic of a map. We will now confide our discussion for the rest of this section to orientable regular maps. Using Equation 5, for orientable map, we have the following result

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{q} = \frac{\chi(X)}{2|E|} = \frac{2 - 2g}{2|E|} = \frac{1 - g}{|E|}$$

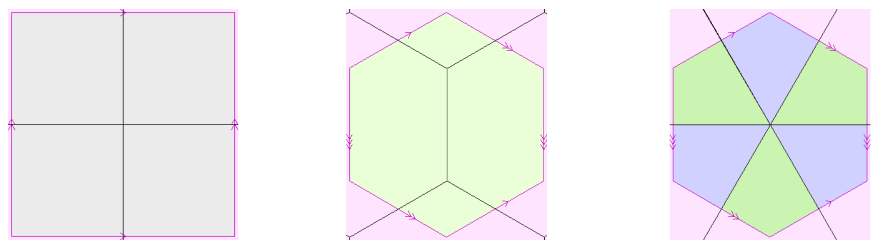
An interesting result is that with small genus, we can find the exact pairs $\{p, q\}$, potentially representing some orientable regular map, that can satisfy this equations. We provide a list of such pairs for $g = 0, 1$, corresponding to $\chi = 2, 0$. Table 1 gives all these regular maps.

It comes at no surprise that the genus-0 regular maps represent the Platonic solids, as shown in Figure 3. The representative corresponding toroidal maps are shown in Figure 10. The word *representative* is used to note that there are different regular maps with the same valency p, q in an infinite family that will later be discussed.

We make a further remark that while this method of examining the possible values for the pair $\{p, q\}$ can assist in narrowing the universe of possible regular maps, not every such map exists. Excluding the cases where $|E|, |V|, |F|$ associated with such pairs take fractional

g	χ	p	q	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert F\rvert$	Commonly known representation
		3	3	6	4	4	Tetrahedron
		3	4	12	6	8	Cube
0	2	4	3	12	8	6	Octahedron
		3	5	30	12	20	Dodecahedron
		5	3	30	20	12	Icosahedron
		4	4				
1	0	3	6				Toroidal map
		6	3				

Table 1: All possible regular maps for genus $0, 1$.



(a) Square Face $\{4, 4\}$ (b) Hexagonal Face $\{6, 3\}$ (c) Triangular Face $\{3, 6\}$

Figure 10: Representative Toroidal Maps, as found in [12].

values or violate other obvious constraints, we can point to a counterexample of a genus-2 "regular maps." The pair $p = 7, q = 3, |E| = 42$ satisfies the equation

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{1-g}{|E|} = -\frac{1}{|E|}$$

Thus, $|F| = 12, |V| = 28$. However, no such regular map exists. We first show a more general proposition and obtain the result as a corollary. The proof is attributable to Wedd [12]. The webpage contains other proofs of non-existence applicable to some postulated regular maps as well as examples of regular maps for small genuses. The version we state modifies upon that in [12]. Before doing so, we state, without proof, the third Sylow theorem.

Theorem 16 (Third Sylow Theorem)**.** *Let* $\mathrm{Syl}_p(\mathrm{Aut}^+ M)$ *be the set of Sylow-p-subgroups of* $\mathrm{Aut}^+ M$*,* $m = |\mathrm{Aut}^+ M : \mathrm{Syl}_p(\mathrm{Aut}^+ M)|$ *be the index of* $\mathrm{Syl}_p(\mathrm{Aut}^+ M)$ *in* $\mathrm{Aut}^+ M$*, and* n_p *be the number of Sylow-p-subgroups. Then* n_p *divides* m *and* $n_p \equiv 1 \mod p$*.*

Proposition 17. *A postulated* $\{P, Q\}$ *regular map with* $|E|$ *edges,* P *is a power of some prime* p*, and* $Q > 2$ *for even* P *and* Q *are relatively prime, then its rotation group* $\mathrm{Aut}^+ M$ *must have more than one Sylow-p-subgroup.*

Proof. Because P, Q are relatively prime, the Sylow-p-subgroups are all the rotations of one face while fixing other faces. It is clear that there are many faces, so there must be more than 1 such subgroup. $\square$

Applying Proposition 17, we show the following.

Corollary 18. *No orientable regular map of genus* 2 *of* 3 *heptagonal (7-gon) faces meeting at each vertex exists.*

Proof. The postulated regular map has 28 vertices, 42 edges, and 12 faces. The order of the rotation subgroup is $|\operatorname{Aut}^+ M| = 2|E| = 84$, so $|\operatorname{Aut}^+ M : \operatorname{Syl}_7(\operatorname{Aut}^+ M)| = 12$. Hence, n_7 must satisfies: n_7 divides 12 and $n_7 \equiv 1 \mod 7$. The only such number is $n_7 = 1$. $\square$

Later on, we will appeal to a different method of showing that an orientable regular map $\{3, 7\}$ of genus 2 does not exist.

3 Generating Regular Maps

3.1 Covering Maps

We begin our discussion by defining what covering maps are. Here, we employ the standard definition. Interested readers can look at Lubkin [9] for a generalized notion of covering. Our definition is adapted from the definition in Lee [8].

Definition 19 (Covering Map and Covering Space). *A covering map is a surjective continuous map $\pi : M \to N$ between connected, locally path-connected spaces with the property that each point of N has a neighborhood U that is evenly covered, meaning that each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π. In particular, the d-covering map is one such that $\pi^{-1}(U) = \bigsqcup_{i=1}^{d} V_i$ includes d components. An alternative term for the components is sheets. M is called the d-covering space of N.*

Thus, if M is a d-covering map of N, then the pre-image of every point in N is exactly d points. The usual example of a covering map is the coiled $\mathbb{R}^1$ onto S^1. However, for our purpose, a discussion of mapping between tori would be more beneficial. We claim that the triple torus is a 2-cover of the double torus. We provide a proof by picture with the aid of Figure 11. The triple torus is separated into four regions by follwoing the cuts in 11(a). The double torus is separated into two regions by following the cuts in Figure 11(b). The top and bottom regions of the triple torus cover the bottom region of the double torus. The middle two regions of the triple torus cover the top region of the double torus. In general, one may show by similar methods that a $(d + 1)$-torus is a d-covering of the double torus.

(a) Triple Torus (b) Double Torus

Figure 11: The cuts are made to identify the covering.

We now describe another kind of covering that is of more interest to regular maps: branched covering.

Definition 20 (Branched Covering). *A continuous map $\pi : M \to N$ is a d-branched covering map if there exists a finite set $J \subset M$ such that $\pi|_{M-J} : M - J \to N - \pi(J)$ is a d-covering map. The values of $\pi(J)$ is naturally forced upon by continuity. $j \in J$ is called a branch point.*

We will describe two branched covers. The first is the double branched cover of the sphere by the torus. We will show a proof by picture in Figure 12.[4] We first cut the torus into

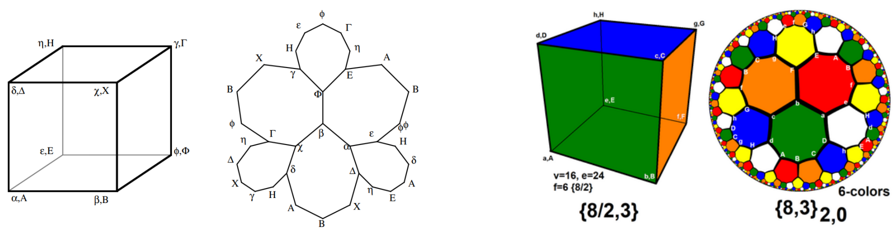
[4]Figure is obtained from the webpage: https://www.researchgate.net/figure/From-left-to-right-the-construction-of-a-double-cover-of-the-sphere-by-the-torus-by_fig10_349336560.

two regions, then transform the region as illustrated until we get the sphere. There are four branch points that is not double covered because for each region and for every hole on the sphere, we need to remove one point to close the hole. Thus, the torus is a double branched cover of the sphere.

Figure 12: Double branched cover of the sphere by the torus.

We have so far found no reference to how these covering on the underlying surface of regular maps can lead to covering of the regular maps themselves by other regular maps. The challenge is of course how the symmetries act differently between maps.

Another example of a branched covering concerns regular maps themselves. This example is due to Grünbaum [6]. Figure 13 illustrates the two maps with the latter represented in two different ways. The branch points are the vertices of the map.



(a) Second map as a 6-face polygon (orginally describe by Grünbaum [6])

(b) Second map as an infinite tesselation of the hyperbolic plane.

Figure 13: Double Cube and An Infinite Tesselation of the Hyperbolic Plane.

3.2 Infinite Family of Regular Maps on the Torus

Recall Equation 5:

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = \frac{\chi(X)}{2|E|}$$

When the underlying surface is the torus, i.e. of genus 1, $\chi = 0$, and the equation becomes

$$\frac{1}{q} - \frac{1}{2} + \frac{1}{p} = 0$$

Note that now, there is no reference to the number of edges, which were previously computable. This issue is unique to genus-0 regular maps. We previously discuss that the only possible pairs are $(4, 4), (3, 6), (6, 3)$, which is noticed by Kepler (as discussed by Coxeter [3]) to represent the infinite tessellation of the plane. We want to explain this relationship with a corollary of the following theorem.

Theorem 21. *Suppose X is a space, G is a group action on X. Let H be subgroup of G with $n = |G : H|$ finite. Let X_K be the orbit of X under the action of group K. Then there exists a homomorphism $\phi : X_H \to X_G$ such that $\phi^{-1}(z)$ has n points, for any $z \in X_G$.*

Proof. Let π_G, π_H be the quotient map that takes X to X_G, X_H, respectively. Define

$$\phi = \pi_G \circ \pi_H^{-1} : X_H \to X_G$$

We check that f is well-defined. Given some $\bar{x}_H = \bar{y}_H$ in X_H, there exists $g \in H \subseteq G$ such that $gx = y$. Thus, $\phi(\bar{x}_H) = \bar{x}_G = \bar{y}_G = \phi(\bar{y}_H)$. We now check that $\phi^{-1}(\bar{x}_G)$ has n points. Let g_iH be distinct left cosets of H ($i = 1, \ldots, n$). Wlog, let $g_1H = H$. Trivially, $\phi(\bar{x}_H) = \bar{x}_G$. Further,

$$\phi(\overline{g_i x_H}) = \pi_G(g_i H x) = \bar{x}_G$$

For any other $g \notin \{g_1, \ldots, g_n\}$, $g \in g_jH$ for some $j \in \{1, \ldots, n\}$. Thus, $\overline{gx_H} = \overline{g_j x_H}$ in X_H, so $\overline{gx_H}$ is not a new distinct point. Therefore, exactly n points are in $\phi^{-1}(z)$ for any $z \in X_G$. $\square$

We will now discuss the tessellation $\{4, 4\}$ to illustrate the application of Theorem 21. Here, we choose a natural choice being that the plane $\mathbb{R}^2$ is tessellated by the square grid whose vertices are $\mathbb{Z}^2$. Then the free Abelian group $\mathbb{Z} \oplus \mathbb{Z}$. This is the simplest identification in Figure 9(a), call this map M. The identification in Figure 9(b), call this map N, is given by the group generated by the translations by $(2, 1)$ and $(-1, 2)$, i.e. the group action is $\langle (2, 1), (-1, 2) \rangle$, which is a (normal) subgroup of index 5 of $\mathbb{Z} \oplus \mathbb{Z}$. By Theorem 21, we find that $|V(N)| = 5|V(M)|$, $|E(N)| = 5|E(M)|$, and $|F(N)| = 5|F(M)|$. Because $M = (1, 2, 1)$, $N = (5, 10, 5)$. Similarly, we can generate an infinite family of $\{4, 4\}$ regular maps on the torus and the same for $\{3, 6\}$ and $\{6, 3\}$.

3.3 Wilson's Operators on Regular Maps

The following section is adapted from the discussion by Wilson [14]. We will discuss the operators: D, P, opp, H_j. We will define them in turn. We first introduce the dual map.

Definition 22 (Dual Map)**.** *The dual of a map $M = (V, E, F)$ is the map $M^* = (V^*, E^*, F^*)$ is the map constructed by the following*

- *In every face, identify the center (or any point) of the face. This set of points becomes the vertex set V^* of the dual map M^*.*
- *Connect two vertices by an edge if they are from adjacent faces in M. This set of edges becomes the edge set E^* of the dual map M^*.*
- *For every vertex in V, let the faces that surround that vertex to be F_v and the dual vertices to those faces be V_v^*. Denote every V_v^*, by permuting the vertices to form a circuit, as a face.*

The existence of the dual map is non-trivial except under the definition of maps by Grünbaum [6], where the axioms are applied equally to both vertices and faces. For the definition by McMullen and Schulte [10], the dual map is achieved by reversing the order of the polyhedron $\mathcal{P}$. We define the operator $D = M \mapsto D(M)$, the dual map of M. An important fact about dual map is that M is regular [reflexible, uniform] iff $D(M)$ is regular [reflexible, uniform]. The proof of this is a consequence of Poincaré duality theorem. A note is that if $D(M) = M$, we call the map self-dual.

Examples of duality can be found in the Platonic solids. The tetrahedron $\{3, 3\}$ is a self-dual, while the cube $\{4, 3\}$ is dual with the octahedron $\{3, 4\}$, and the icosahedron $\{3, 5\}$ is dual with the dodecahedron $\{5, 3\}$. This is reflected in their Schläfli symbols. In particular, the dual of the map $\{p, q\}$ is the map $\{q, p\}$.

The next two operators require first defining the concept of holes and Petrie paths.

Definition 23 (jth holes and Petrie paths – Wilson [14])**.** *A jth order hole is a cyclic sequence of edges, each two consecutive sharing a vertex, so that at each vertex, the adjacent edges subtend j faces on on one side, either the right or the left but consistently throughout.*

A jth order Petrie path is a similar sequence of edges, but at each vertex, j faces are enclosed on the right and on the left alternately.

A first-order Petrie path is called a Petrie path and a first-order hole is just a face.

We now ready to define the second operator, the Petrie dual. However, as Wilson noted, this operator is not closed for chiral maps. As such, for the remainder of our discussion, we will refer only to reflexible map when discussing regular maps. Another note is that the concept of Petrie dual was not due to Wilson, as the name suggests. It was introduced by Coxeter in [3] through his discussion and work with Petrie. Wilson defines the Petrie operator as follows.

Definition 24 (Petrie map)**.** *The Petrie of a map M is the map $P(M)$ constructed by dissolving the faces of M and span by a membrane each cycle of edges which forms a Petrie path in M. The resulting figure is a map on a surface, in general a different surface than that of M. A set of edges which forms a face in M forms a Petrie path in $P(M)$, while vertices in M are also vertices of $P(M)$.*

Similar to (Poincaré) duality, for reflexible maps, M is regular iff its Petrie dual $P(M)$ is regular. For chiral map, Wilson states that if M is regular then $P(M)$ must be uniform but not regular. P and D satisfy

$$I = P^2 = D^2 = (PD)^3$$

which means $\langle P, D\rangle \approx S_3$, the permutation group. We now define the third operation, $\mathrm{opp}(M) = DPD(M) = PDP(M)$, called the opposite operator. Wilson defines it explicitly as follows.

Definition 25. *Given a regular map M, obtain* $\mathrm{opp}(M)$ *as follows*

1. *Label each edge with a number and an arrow running along it on both sides,*
2. *Cut the map apart along the edges, and then*
3. *Glue it back together again so that all the numbers match but none of the arrows do.*

Its faces are the faces of M, but all of the joinings have been reversed.

The relationship between these three operations can be visualized by the effect on edges across two darts. Figure 14 is taken from [14], where Wilson demonstrate this relationship. Here, the faces (..125..) and (..324..), which meet at 2 in M and also in $\mathrm{opp}(M) = PDP(M)$ but are oppositely matched.

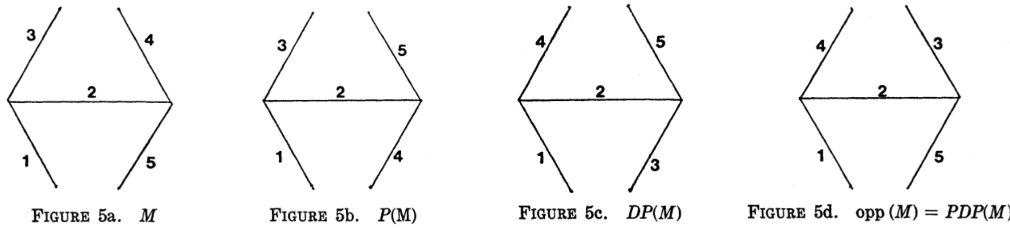


Figure 14: Wilson's operators D, P, and opp as demonstrated in [14].

Finally, let us introduce the H_j operation by quoting Wilson [14].

> Consider the object we get by dissolving the faces of M and spanning by a membrane each cycle of edges which is a jth order hole in M. If j is relatively prime to the valence q of a vertex, this will be a map on a manifold which we can call $H_j(M)$. If however, d = (i, q) is not 1, the jth order holes meeting at a given vertex will resolve themselves into d cycles of q/d holes apiece, none from one cycle meeting one from any other (at that vertex). Thus our putative $H_j(M)$ will look like a manifold except that at each vertex, d sheets will be pinched together. In this case, we separate each vertex into (j, q) vertices, one on each sheet. The result of this surgery is a manifold which is either one connected map or the union of a number of identical connected components. $H_j(M)$ is one connected component of this manifold. Clearly $H_iH_j = H_{ij}$ and $PH_i = H_iP$

Coxeter introduced jth order holes. While no one seems previously to have constructed an operation like the H_j's, the idea also is not new: the Great Dodecahedron is formed precisely by making H_2 of the icosahedron. Further, the Great Icosahedron may be seen to be H_2 of the Small Stellated Dodecahedron.

The benefit of using these operators allows us to construct new maps that are regular when the base map is regular. The close associations between these maps also allow for a study of one map to inform properties of the others.

3.4 Conder and Dobcsányi 's Automorphism Group Approach

The following section is adapted from the discussion by Conder and Dobcsányi [1]. We begin by describing the relationship between genus and the automorphism group. Because regular maps are flag-transitive, by definition, the rotation subgroup is of the order of the number of flags in the map. Given that every edge is represented by two flags, we obtain $|\operatorname{Aut}^+ M| = 2|E|$. Thus, for reflexible regular maps, $|\operatorname{Aut} M| = 2|\operatorname{Aut}^+ M| = 4|E|$ and for chiral regular maps, $|\operatorname{Aut} M| = |\operatorname{Aut}^+ M| = 2|E|$. Hence, we obtain the following relationship, as in [1].

$$g = g(M) = \begin{cases} |\operatorname{Aut} M|(1/8 - 1/4p - 1/4q) + 1 & \text{if } M \text{ is orientable and reflexible} \\ |\operatorname{Aut} M|(1/4 - 1/2p - 1/2q) + 1 & \text{if } M \text{ is orientable and chiral} \\ |\operatorname{Aut} M|(1/4 - 1/2p - 1/2q) + 2 & \text{if } M \text{ is non-orientable} \end{cases} \tag{6}$$

We further link the rotation subgroup with the triangle group. For a regular map $\{p, q\}$, $\Delta = \Delta(p, q, 2) = \langle u, v : u^p = v^q = (uv)^2 = 1\rangle \cong \operatorname{Aut}^+ M$ via some non-degenerate homomorphism ϕ such that $\phi(u) = R, \phi(v) = S$. The converse says that for any non-degenerate homomorphism θ from $\Delta(p, q, 2)$ to a finite group G, we have a $\{p, q\}$ regular map M. In particular, let $R = \theta(u), S = \theta(v)$, then vertex, edge, and face sets are (right) cosets of the subgroups $V = \langle S\rangle, E = \langle RS\rangle, F = \langle R\rangle$ in G.

For reflexible maps, our previous discussion on automorphisms allows us to check if one kind of reflection is in the automorphism group for all three reflections to be in the group. Thus, we examine a second triangle group $\Delta^* = \Delta^*(p, q, 2) = \langle t, u, v : t^2 = (ut)^2 = (tv)^2 = u^p = v^q = (uv)^2 = 1\rangle$ to impose the three reflections. If a, b, and c are images of ut, t, tv, then the vertex, edge, and face sets are cosets of the subgroups $V = \langle b, c\rangle, E = \langle a, c\rangle, F = \langle a, b\rangle$ in G. The map is orientable iff $\langle ab, bc\rangle$ is a subgroup of index 2.

We have now changed the problem from determining the regular map (up to a specific genus) into determining all non-degenerate finite homomorphic images of the groups $\Delta(p, q, 2)$ and $\Delta^*(p, q, 2)$. Conder and Dobcsányi made another simplification by instead of considering all possible pairs $\{p, q\}$, they consider the two more general groups of these triangle groups: $\Delta < \Phi = \langle u, v : (uv)^2 = 1\rangle$ and $\Delta^* < \Phi^* = \langle t, u, v : t^2 = (ut)^2 = (tv)^2 = (uv)^2 = 1\rangle$. The orders of the images of u and v in G will be p and q, respectively. We can then make further classification to the kind of regular map obtained.

- We first consider the homomorphic images G of Φ^*, which are reflexible maps.
  - This map M is non-orientable if and only if the image of t lies in the subgroup generated by the images of u and v, or equivalently, since t inverts each of u and v by conjugation, if and only if there exists some relation involving the images of u, v and t in which the number of occurrences of t is odd.
  - Otherwise (when there is no such relation) the map is orientable and reflexible.
- We can then consider the homorphic images G of Φ but not those of Φ^* which forms our orientable and chiral regular maps. Or more directly, for such a map, there exists no automorphism of G which inverts the images of each of u and v by conjugation.

The genus can then be calculated according Equation 6. For computational purposes, a cutoff genus provides a cap on $|G| = |\operatorname{Aut} M|$. Of course, if one is only interested in the existence a $\{p, q\}$ regular map, the same procedure can be used on Δ, Δ^* rather than Φ, Φ^*. For further discussion on the computational algorithm and a list of all regular maps determined for genus up to 15 for orientable regular maps and 30 for non-orientable regular maps, we refer to Conder and Dobcsányi [1] and Dobcsányi [5].

Finally, by referring to the list by Conder and Dobcsányi we find no orientable $\{7, 3\}$-regular map of genus 2. Because the list is exhaustive of all orientable regular maps of genus 2, we have a different way of showing that this map does not exist.

Acknowledgement

I want to thank Professor Douglas Ravenel for introducing me to the study regular maps, his continued support and mentorship, and his long interests in polyhedra. This thesis would not have begun without the support of Professor Ravenel. I also want to thank Professor Jonathan Pakianathan for his classes – Undergraduate Algebra and Topology, and Computational Geometry – as they continue to inform my understanding of algebraic geometry. Finally, this work is a survey of the literature, so I am in debt of their work and continued contribution to this subject.

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