ANNIHILATING PAIRS INEQUALITIES IN ORLICZ SPACES

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Abstract. Some results about annihilating pairs.

Contents

1. Introduction

In this paper we generalize a result from Jaming, Iosevich, and Mayeli [2] to Orlicz spaces.

2. Fourier Preliminaries and Previous Results

First, we introduce some definitions and basic theorems relating to Fourier transforms in L^p spaces.

Definition 2.1. Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. The **Fourier transform** of f is given by

$$\hat{f}(m) = N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x),$$

where $\chi(t) = e^{\frac{2\pi i t}{N}}$. The corresponding **inverse Fourier transform** is given by

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m).$$

We may verify that these are indeed inverse processes, as

$$\begin{aligned} N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m) &= N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \left(N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^d} \chi(-y \cdot m) f(y) \right) \\ &= N^{-d} \sum_{m \in \mathbb{Z}_N^d} \sum_{y \in \mathbb{Z}_N^d} \chi(x - y \cdot m) f(y) \\ &= N^{-d} \sum_{y \in \mathbb{Z}_N^d} f(y) \sum_{m \in \mathbb{Z}_N^d} \chi(x - y \cdot m) \end{aligned}$$

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$$= f(x),$$

since if $x \neq y$, the Gauss sum $\sum_{m\in\mathbb{Z}_N^d} \chi((x-y)\cdot m) = 0$, whereas $\sum_{m\in\mathbb{Z}_N^d} \chi(0) = N^d$ when $x = y$.

Definition 2.2. An $\mathbf{L^p}$ **space** over $\mathbb{Z}_N^d$ is the space

$$L^p(\mathbb{Z}_N^d) = \left\{ f : \mathbb{Z}_N^d \to \mathbb{C} \;\middle|\; \left(\sum_{x\in\mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}} < \infty \right\},$$

for $p \in [1, \infty)$, or

$$L^\infty(\mathbb{Z}_N^d) = \left\{ f : \mathbb{Z}_N^d \to \mathbb{C} \;\middle|\; \max_{x\in\mathbb{Z}_N^d} |f(x)| < \infty \right\}$$

for $p = \infty$. We will consider the following norms with respect to sets $A \subseteq \mathbb{Z}_N^d$ and functions f in the L^p space: The $\mathbf{L^p}$ **norm**

$$\|f\|_{L^p(A)} = \left(\sum_{x\in A} |f(x)|^p \right)^{\frac{1}{p}} \qquad \|f\|_{L^\infty(A)} = \max_{x\in A} |f(x)|,$$

and the **normalized** $\mathbf{L^p}$ **norm**

$$\|f\|_{L^p(\mu_A)} = \left(\frac{1}{|A|} \sum_{x\in A} |f(x)|^p \right)^{\frac{1}{p}} \qquad \|f\|_{L^\infty(\mu_A)} = \max_{x\in A} |f(x)|.$$

Also, we will use the shorthand $\|f\|_{L^p(\mu)} := \|f\|_{L^p(\mu_{\mathbb{Z}_N^d})}$.

Theorem 2.3 (Plancherel)**.** *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Then*

$$\|\hat{f}\|_{L^2(\mathbb{Z}_N^d)} = \|f\|_{L^2(\mathbb{Z}_N^d)}.$$

Proof. Squaring the left hand side and expanding, we obtain

$$\begin{aligned}
\|\hat{f}\|^2_{L^2(\mathbb{Z}_N^d)} &= \sum_{m\in\mathbb{Z}_N^d} |\hat{f}(m)|^2 \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d} \left| \sum_{x\in\mathbb{Z}_N^d} \chi(-x\cdot m) f(x) \right|^2 \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d} \left(\overline{\sum_{x\in\mathbb{Z}_N^d} \chi(-x\cdot m) f(x)} \right) \left(\sum_{x\in\mathbb{Z}_N^d} \chi(-x\cdot m) f(x) \right) \\
&= \sum_{m\in\mathbb{Z}_N^d} N^{-d} \sum_{x,y\in\mathbb{Z}_N^d} \chi((x-y)\cdot m) \overline{f(x)} f(y) \\
&= \sum_{x,y\in\mathbb{Z}_N^d} \overline{f(x)} f(y) N^{-d} \sum_{m\in\mathbb{Z}_N^d} \chi((x-y)\cdot m).
\end{aligned}$$

Note now that for any $x \neq y$, the Gauss sum $\sum_{m\in\mathbb{Z}_N^d} \chi((x-y)\cdot m) = 0$, whereas $\sum_{m\in\mathbb{Z}_N^d} \chi(0) = N^d$ when $x = y$, hence

$$\|\hat{f}\|^2_{L^2(\mathbb{Z}_N^d)} = \sum_{x\in\mathbb{Z}_N^d} |f(x)|^2 = \|f\|^2_{L^2(\mathbb{Z}_N^d)},$$

as desired. □

Theorem 2.4 (Riesz-Thorin)**.** *Let* $p_0, p_1, q_0, q_1 \in [1, \infty]$*. For* $t \in (0, 1)$*, let*

$$p_t = \frac{1}{\frac{1-t}{p_0} + \frac{t}{p_1}} \qquad q_t = \frac{1}{\frac{1-t}{q_0} + \frac{t}{q_1}}.$$

Then if $T : (L^{p_0} + L^{p_1})(\mathbb{Z}_N^d) \to (L^{q_0} + L^{q_1})(\mathbb{Z}_N^d)$ *is a linear operator such that for* $f_0 \in L^{p_0}(\mathbb{Z}_N^d)$ *and* $f_1 \in L^{p_1}(\mathbb{Z}_N^d)$*,*

$$\|Tf_0\|_{L^{q_0}(\mathbb{Z}_N^d)} \leq M_0 \|f_0\|_{L^{p_0}(\mathbb{Z}_N^d)} \qquad \|Tf_1\|_{L^{q_1}(\mathbb{Z}_N^d)} \leq M \|f_1\|_{L^{p_1}(\mathbb{Z}_N^d)},$$

we have that

$$\|Tf\|_{L^{q_t}(\mathbb{Z}_N^d)} \leq M_0^{1-t} M_1^t \|f\|_{L^{p_t}(\mathbb{Z}_N^d)}$$

for $f \in L^{p_t}(\mathbb{Z}_N^d)$*.*

Theorem 2.5 (Hausdorff-Young)**.** *If* $f : \mathbb{Z}_N^d \to \mathbb{C}$ *and* $1 \leq p \leq 2$*,*

$$\|\hat{f}\|_{L^{p'}(\mathbb{Z}_N^d)} \leq N^{-\frac{d}{2}(\frac{2-p}{p})} \|f\|_{L^p(\mathbb{Z}_N^d)},$$

where $p' = \frac{1}{1-\frac{1}{p}}$*.*

Proof. In the $p = 1$ case, we have

$$\begin{aligned} \|\hat{f}\|_\infty &= \max_{m \in \mathbb{Z}_N^d} \left| N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x) \right| \\ &\leq N^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}_N^d} |f(x)| \\ &= N^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{Z}_N^d)}. \end{aligned}$$

In the $p = 2$ case, we have

$$\|\hat{f}\|_{L^2(\mathbb{Z}_N^d)} = \|f\|_{L^2(\mathbb{Z}_N^d)}$$

by Theorem 2.3. Now, letting $p_0 = \infty$, $p_1 = 2$, $q_0 = 1$, $q_1 = 2$, we see that the Fourier transform operator satisfies the assumptions of Theorem 2.4, hence

$$\|\hat{f}\|_{L^{p'}(\mathbb{Z}_N^d)} \leq N^{-\frac{d}{2}(\frac{2-p}{p})} \|f\|_{L^p(\mathbb{Z}_N^d)}$$

for all $p \in [1, 2]$. □

We now prove some previous results regarding annihilating pairs inequalities in $\mathbb{Z}_N^d$.

Theorem 2.6 (Ghobber and Jaming [1])**.** *Let* $f : \mathbb{Z}_N^d \to \mathbb{C}$*. If* $E, S \subset \mathbb{Z}_N^d$*,* $|E||S| < N^d$*, then*

$$\|f\|_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}}\right) \left(\|f\|_{L^2(E^{\complement})} + \|\hat{f}\|_{L^2(S^{\complement})}\right).$$

Proof. Consider

$$\begin{aligned}
\|\widehat{1_E f}\|_{L^2(S)} &= \left(\sum_{m\in S} |\widehat{1_E f(m)}|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{m\in S} \left| N^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}_N^d} \chi(-x\cdot m) 1_E f(x)\right|^2\right)^{\frac{1}{2}} \\
&\le N^{-\frac{d}{2}} \left(\sum_{m\in S}\left(\sum_{x\in E} |\chi(-x\cdot m) f(x)|\right)^2\right)^{\frac{1}{2}} \\
&= N^{-\frac{d}{2}} \left(\sum_{m\in S}\left(\sum_{x\in E} |f(x)|\right)^2\right)^{\frac{1}{2}} \\
&= N^{-\frac{d}{2}} \left(\sum_{m\in S} \|f\|_{L^1(E)}^2\right)^{\frac{1}{2}} \\
&= N^{-\frac{d}{2}} |S|^{\frac{1}{2}} \|f\|_{L^1(E)} \\
&\le N^{-\frac{d}{2}} |S|^{\frac{1}{2}} |E|^{\frac{1}{2}} \|f\|_{L^2(E)},
\end{aligned}$$

where the last inequality follows by Cauchy-Schwarz. Now, via triangle inequality and Theorem 2.3,

$$\begin{aligned}
\|\widehat{1_E f}\|_{L^2(S^\complement)} &\ge \|\widehat{1_E f}\|_{L^2(\mathbb{Z}_N^d)} - \|\widehat{1_E f}\|_{L^2(S)} \\
&\ge \|f\|_{L^2(E)} \left(1 - N^{-\frac{d}{2}} |S|^{\frac{1}{2}} |E|^{\frac{1}{2}}\right),
\end{aligned}$$

hence

$$\begin{aligned}
\|f\|_{L^2(\mathbb{Z}_N^d)} &\le \|f\|_{L^2(E)} + \|f\|_{L^2(E^\complement)} \\
&\le \frac{\|\widehat{1_E f}\|_{L^2(S^\complement)}}{1 - \sqrt{\frac{|E||S|}{N^d}}} + \|f\|_{L^2(E^\complement)} \\
&= \frac{\|\hat{f} - \widehat{1_{E^\complement} f}\|_{L^2(S^\complement)}}{1 - \sqrt{\frac{|E||S|}{N^d}}} + \|f\|_{L^2(E^\complement)} \\
&= \frac{\|\hat{f}\|_{L^2(S^\complement)} + \|f\|_{L^2(E^\complement)}}{1 - \sqrt{\frac{|E||S|}{N^d}}} + \|f\|_{L^2(E^\complement)} \\
&\le \left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}}\right) \left(\|f\|_{L^2(E^\complement)} + \|\hat{f}\|_{L^2(S^\complement)}\right).
\end{aligned}$$

□

Under certain assumptions, there is an improvement to the above result which is of interest. With that in mind, we introduce the following definition:

Definition 2.7. We say that $S \subset \mathbb{Z}_N^d$ satisfies a (p,q) **Fourier restriction estimate** $(1 \le p \le q)$ with uniform constant $C_{p,q} > 0$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|} \sum_{m \in S} |\hat{f}(m)|^q \right)^{\frac{1}{q}} \le C_{p,q} N^{-\frac{d}{2}} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

Theorem 2.8 (Jaming, Iosevich, and Mayeli [2])**.** *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$, $E, S \subset \mathbb{Z}_N^d$, and suppose that a (p,q) Fourier restriction estimate holds for S with constant $C_{p,q}$. If $1 \le p \le 2 \le q$ and $|E|^{\frac{2-p}{p}}|S| < \frac{N^d}{C_{p,q}^2}$, then*

$$\|f\|_{L^2(\mathbb{Z}_N^d)} \le \left(1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}} \right) \left(\|f\|_{L^2(E^{\complement})} + \|\hat{f}\|_{L^2(S^{\complement})} \right).$$

If $1 \le p \le q \le 2$ and $|E|^{\frac{q-p}{p}}|S| < \frac{N^d}{C_{p,q}^q}$, then

$$\|f\|_{L^2(\mathbb{Z}_N^d)} \le \left(1 + \frac{|E|^{\frac{1}{2}-\frac{1}{q}}}{1 - \sqrt[q]{\frac{C_{p,q}^q |E|^{\frac{q-p}{p}} |S|}{N^d}}} \right) \left(\|f\|_{L^2(E^{\complement})} + \|\hat{f}\|_{L^2(S^{\complement})} \right).$$

Proof. Consider first the $1 \le p \le 2 \le q$ case. By the restriction assumption,

$$\begin{aligned}
\|\widehat{1_E f}\|_{L^2(S)} &= |S|^{\frac{1}{2}} \|\widehat{1_E f}\|_{L^2(\mu_S)} \\
&\le |S|^{\frac{1}{2}} \|\widehat{1_E f}\|_{L^q(\mu_S)} \\
&\le C_{p,q} |S|^{\frac{1}{2}} N^{-\frac{d}{2}} \|f\|_{L^p(E)} \\
&= C_{p,q} |S|^{\frac{1}{2}} N^{-\frac{d}{2}} \left(\|f^p\|_{L^1(E)} \right)^{\frac{1}{p}}.
\end{aligned}$$

By Hölder's inequality, this quantity is bounded by

$$\begin{aligned}
C_{p,q}|S|^{\frac{1}{2}} N^{-\frac{d}{2}} \left(\|f^p\|_{L^{\frac{2}{p}}(E)} |E|^{\frac{2-p}{2}} \right)^{\frac{1}{p}} &= C_{p,q}|S|^{\frac{1}{2}} N^{-\frac{d}{2}} |E|^{\frac{2-p}{2p}} \|f\|_{L^2(E)} \\
&= \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}} \|f\|_{L^2(E)}.
\end{aligned}$$

Now, via triangle inequality and Theorem 2.3,

$$\begin{aligned}
\|\widehat{1_E f}\|_{L^2(S^{\complement})} &\ge \|\widehat{1_E f}\|_{L^2(\mathbb{Z}_N^d)} - \|\widehat{1_E f}\|_{L^2(S)} \\
&\ge \|f\|_{L^2(E)} \left(1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}} \right),
\end{aligned}$$

hence

$$\begin{aligned}
\|f\|_{L^2(\mathbb{Z}_N^d)} &\le \|f\|_{L^2(E)} + \|f\|_{L^2(E^{\complement})} \\
&\le \frac{\|\widehat{1_E f}\|_{L^2(S^{\complement})}}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}} + \|f\|_{L^2(E^{\complement})}
\end{aligned}$$

$$= \frac{\|\hat{f} - \widehat{1_{E^\complement} f}\|_{L^2(S^\complement)}}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}} + \|f\|_{L^2(E^\complement)}$$

$$= \frac{\|\hat{f}\|_{L^2(S^\complement)} + \|f\|_{L^2(E^\complement)}}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}} + \|f\|_{L^2(E^\complement)}$$

$$\leq \left(1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}}\right) \left(\|f\|_{L^2(E^\complement)} + \|\hat{f}\|_{L^2(S^\complement)}\right).$$

Consider now the $1 \leq p \leq q \leq 2$ case. By the restriction assumption and Hölder's inequality,

$$\begin{aligned} \|\widehat{1_E f}\|_{L^q(S)} &= S^{\frac{1}{q}} \|\widehat{1_E f}\|_{L^q(\mu_S)} \\ &\leq C_{p,q} |S|^{\frac{1}{q}} N^{-\frac{d}{2}} \|f\|_{L^p(E)} \\ &\leq C_{p,q} |S|^{\frac{1}{q}} |E|^{\frac{1}{p} - \frac{1}{2}} N^{-\frac{d}{2}} \|f\|_{L^2(E)}. \end{aligned}$$

Now, by Theorem 2.5, we have

$$\begin{aligned} \|\widehat{1_E f}\|_{L^q(\mathbb{Z}_N^d)} &\geq N^{\frac{d}{2}(\frac{2-q}{q})} \|f\|_{L^{q'}(E)} \\ &= N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{2} - \frac{1}{q'}} \|f\|_{L^2(E)} \\ &= N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} \|f\|_{L^2(E)}, \end{aligned}$$

where $q' = \frac{1}{1 - \frac{1}{q}}$. With triangle inequality, we combine the above, obtaining

$$\begin{aligned} \|\widehat{1_E f}\|_{L^q(S^\complement)} &\geq \|\widehat{1_E f}\|_{L^q(\mathbb{Z}_N^d)} - \|\widehat{1_E f}\|_{L^q(S)} \\ &\geq \left(N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} - |S|^{\frac{1}{q}} |E|^{\frac{1}{p} - \frac{1}{2}} C_{p,q} N^{-\frac{d}{2}}\right) \|f\|_{L^2(E)} \\ &\geq N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} \left(1 - |S|^{\frac{1}{q}} |E|^{\frac{1}{p} - \frac{1}{q}} C_{p,q} N^{-\frac{d}{q}}\right) \|f\|_{L^2(E)} \\ &\geq N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} \left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}} C_{p,q}^q}{N^d}}\right) \|f\|_{L^2(E)} \end{aligned}$$

Now

$$\begin{aligned} \|f\|_{L^2(\mathbb{Z}_N^d)} &\leq \|f\|_{L^2(E)} + \|f\|_{L^2(E^\complement)} \\ &\leq \frac{\|\widehat{1_E f}\|_{L^q(S^\complement)}}{N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} \left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}} C_{p,q}^q}{N^d}}\right) \|f\|_{L^2(E)}} + \|f\|_{L^2(E^\complement)} \\ &= \frac{\|\hat{f} - \widehat{1_{E^\complement} f}\|_{L^q(S^\complement)}}{N^{d(\frac{1}{q} - \frac{1}{2})} |E|^{\frac{1}{q} - \frac{1}{2}} \left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}} C_{p,q}^q}{N^d}}\right) \|f\|_{L^2(E)}} + \|f\|_{L^2(E^\complement)} \end{aligned}$$

$$= \frac{\|\hat{f}\|_{L^q(S^{\complement})} + \|f\|_{L^q(E^{\complement})}}{N^{d(\frac{1}{q}-\frac{1}{2})}|E|^{\frac{1}{q}-\frac{1}{2}}\left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}}C_{p,q}^q}{N^d}}\right)\|f\|_{L^2(E)}} + \|f\|_{L^2(E^{\complement})}$$

$$\leq \frac{|S^{\complement}|^{\frac{1}{q}-\frac{1}{2}}\|\hat{f}\|_{L^2(S^{\complement})} + \|f\|_{L^2(E^{\complement})}}{N^{d(\frac{1}{q}-\frac{1}{2})}|E|^{\frac{1}{q}-\frac{1}{2}}\left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}}C_{p,q}^q}{N^d}}\right)\|f\|_{L^2(E)}} + \|f\|_{L^2(E^{\complement})}$$

$$\leq \frac{\|\hat{f}\|_{L^2(S^{\complement})} + \|f\|_{L^2(E^{\complement})}}{|E|^{\frac{1}{q}-\frac{1}{2}}\left(1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}}C_{p,q}^q}{N^d}}\right)\|f\|_{L^2(E)}} + \|f\|_{L^2(E^{\complement})}$$

$$\leq \left(1 + \frac{|E|^{\frac{1}{2}-\frac{1}{q}}}{1 - \sqrt[q]{\frac{|S||E|^{\frac{q-p}{p}}C_{p,q}^q}{N^d}}}\right)\left(\|f\|_{L^2(E^{\complement})} + \|\hat{f}\|_{L^2(S^{\complement})}\right).$$

□

3. Orlicz Space Preliminaries

Orlicz spaces are a natural generalization of L^p spaces.

Definition 3.1. A function $\Phi : [0,\infty) \to [0,\infty]$ is a **Young function** if it is convex and $\lim_{x\to\infty}\Phi(x) = 0$, $\lim_{x\to\infty}\Phi(x) = \infty$. The **complementary Young function** to Φ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) \mid x \geq 0\}, y \in [0,\infty).$$

The **inverse** to a Young function Φ is defined by

$$\Phi^{-1}(x) = \inf\{y \geq 0 \mid \Phi(y) > x\}.$$

Finally, a Young function is called **nice** if it satisfies both $\lim_{x\to 0}\frac{\Phi(x)}{x} = 0$ and $\lim_{x\to\infty}\frac{\Phi(x)}{x} = \infty$.

Note that a pair of complementary Young functions (Φ,Ψ) satisfies Young's inequality

$$|xy| \leq \Phi(x) + \Psi(y).$$

Definition 3.2. An **Orlicz space** over $\mathbb{Z}_N^d$ is the space

$$L^{\Phi}(\mathbb{Z}_N^d) = \left\{ f : \mathbb{Z}_N^d \to \mathbb{C} \;\middle|\; \sum_{x\in\mathbb{Z}_N^d}\Phi\left(\frac{|f(x)|}{k}\right) \leq 1 \right\}.$$

Let Ψ denote the complementary Young function to Φ. We will consider the following norms with respect to sets $A \subseteq \mathbb{Z}_N^d$ and functions f in the Orlicz space: The **Orlicz norm**

$$\|f\|_{L^{(\Phi)}(A)} = \sup\left\{ \sum_{x\in A}|f(x)g(x)| \;\middle|\; \sum_{x\in A}\Psi(|g(x)|) \leq 1 \right\},$$

the **normalized Orlicz norm**

$$\|f\|_{L^{(\Phi)}(A)} = \sup\left\{ \frac{1}{|A|}\sum_{x\in A}|f(x)g(x)| \;\middle|\; \frac{1}{|A|}\sum_{x\in A}\Psi(|g(x)|) \leq 1 \right\},$$

the **Luxemburg norm**

$$\|f\|_{L^{\Phi}(A)} = \inf\left\{ k>0 \;\middle|\; \sum_{x\in A}\Phi\left(\frac{|f(x)|}{k}\right) \leq 1 \right\},$$

and the **normalized Luxemburg norm**

$$\|f\|_{L^{\Phi}(\mu_A)} = \inf\left\{ k>0 \;\middle|\; \frac{1}{|A|}\sum_{x\in A}\Phi\left(\frac{|f(x)|}{k}\right) \leq 1 \right\}.$$

Also, we will use the shorthand $\|f\|_{L^{\Phi}(\mu)} := \|f\|_{L^{\Phi}(\mu_{\mathbb{Z}_N^d})}$.

We now state some theorems in Orlicz spaces, beginning with Hölder's inequality. In fact, a version of this inequality follows very naturally from the definition of our norms.

Theorem 3.3 (Hölder's Inequality [4])**.** *If Φ, Ψ are complementary nice Young functions, for $f \in L^{\Phi}$, $g \in L^{\Psi}$, $A \subseteq \mathbb{Z}_N^d$, we have*

$$\|fg\|_{L^1(\mu_A)} \leq \|f\|_{L^{(\Phi)}(\mu_A)}\|g\|_{L^{\Psi}(\mu_A)}.$$

Proof. Take some $u > \|g\|_{L^{\Psi}(\mu_A)}$. We have that

$$\begin{aligned}\|fg\|_{L^1(\mu_A)} &= \frac{1}{|A|}\sum_{x\in A}|f(x)g(x)| \\ &= \frac{u}{|A|}\sum_{x\in A}|f(x)|\frac{|g(x)|}{u}.\end{aligned}$$

By definition of the Luxemburg norm, $\frac{1}{|A|}\sum_{x\in A}\Psi\left(\frac{|g(x)|}{u}\right) \leq 1$, so by definition of of the Orlicz norm,

$$\frac{1}{|A|}\sum_{x\in A}|f(x)|\frac{|g(x)|}{u} \leq \|f\|_{L^{(\Phi)}(\mu_A)}.$$

Taking $u \to \|g\|_{L^{\Psi}(\mu_A)}$, we obtain

$$\|fg\|_{L^1(\mu_A)} \leq \|f\|_{L^{(\Phi)}(\mu_A)}\|g\|_{L^{\Psi}(\mu_A)},$$

as desired. □

We will also introduce a number of useful lemmas, beginning with the following exact computation of the normalized Orlicz norm of an indicator function.

Lemma 3.4. *Let Φ be a nice Young function with complementary Young function Ψ, and take some $A \subseteq \mathbb{Z}_N^d$. Then*

$$\|1_A\|_{L^{(\Phi)}(\mu)} = \frac{|A|}{N^d}\Psi^{-1}\left(\frac{N^d}{|A|}\right)$$

Proof. Let $g : \mathbb{Z}_N^d \to \mathbb{C}$ satisfy

$$\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} \Psi(|g(x)|) \leq 1.$$

Then, by Jensen's inequality,

$$\Psi\left(\frac{\sum_{x\in A}|g(x)|}{|A|}\right) \leq \frac{1}{|A|}\sum_{x\in A}\Psi(|g(x)|) \leq \frac{N^d}{|A|}.$$

Applying Ψ^{-1} to both sides, we have

$$\sum_{x\in A}|g(x)| \leq |A|\Psi^{-1}\left(\frac{N^d}{|A|}\right),$$

thus

$$\begin{aligned}
\|1_A\|_{L^{(\Phi)}(\mu)} &= \sup\left\{\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|1_A(x)g(x)| \;\middle|\; \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}\Psi(|g(x)|)\leq 1\right\} \\
&= \sup\left\{\frac{1}{N^d}\sum_{x\in A}|g(x)| \;\middle|\; \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}\Psi(|g(x)|)\leq 1\right\} \\
&\leq \frac{|A|}{N^d}\Psi^{-1}\left(\frac{N^d}{|A|}\right).
\end{aligned}$$

Conversely, if $g = \Psi^{-1}\left(\frac{N^d}{|A|}\right)1_A$, then

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}\Psi(|g(x)|) = \frac{1}{N^d}\sum_{x\in A}\frac{N^d}{|A|} = 1,$$

hence

$$\|1_A\|_{L^{(\Phi)}(\mu)} \geq \frac{1}{N^d}\sum_{x\in\mathbb{Z}_N^d}|1_A(x)g(x)| = \frac{|A|}{N^d}\Psi^{-1}\left(\frac{N^d}{|A|}\right).$$

Since both inequalities have been shown, we have that

$$\|1_A\|_{L^{(\Phi)}(\mu)} = \frac{|A|}{N^d}\Psi^{-1}\left(\frac{N^d}{|A|}\right),$$

as desired.

Lemma 3.5. *If Φ is a nice Young function satisfying $x^p \prec \Phi$ for $p \in [1,\infty)$, then for $A \subseteq \mathbb{Z}_N^d$,*

$$\|f\|_{L^p(\mu_A)} \leq \left(\frac{|A|}{N^d}\right)^{\frac{1}{p}}\left(\Phi^{-1}\left(\frac{N^d}{|A|}\right)\right)\|f\|_{L^\Phi(\mu_A)}.$$

Proof. Let $\Psi = \Phi(x^{\frac{1}{p}})$. Since $x^p \prec \Phi$, Ψ is a nice Young function. By Theorem 3.3,

$$\|f\|_{L^p(\mu_A)}^p = \|f^p\|_{L^1(\mu_A)} \leq \|f^p\|_{L^\Psi(\mu_A)}\|1\|_{L^{(\Psi^*)}(\mu_A)},$$

where Ψ^* is the complementary Young function to Ψ. Take some $u > \|f\|_{L^\Phi(\mu_A)}$, so that

$$\frac{1}{|A|}\sum_{x\in A}\Psi\left(\frac{|f^p(x)|}{u^p}\right) = \frac{1}{|A|}\sum_{x\in A}\Phi\left(\frac{|f(x)|}{u}\right) \le 1.$$

Then, letting $u \to \|f\|_{L^\Phi(A)}$, we have that $\|f^p\|_{L^\Psi(\mu_A)} \le u^p = \|f\|^p_{L^\Phi(\mu_A)}$. As such,

$$\|f\|_{L^p(\mu_A)} \le \|f\|_{L^\Phi(\mu_A)}\|1\|^{\frac{1}{p}}_{L^{(\Psi^*)}(\mu_A)}.$$

By Lemma 3.4,

$$\|1\|_{L^{(\Psi^*)}(\mu_A)} = \frac{|A|}{N^d}\Psi^{-1}\left(\frac{N^d}{|A|}\right) = \frac{|A|}{N^d}\left(\Phi^{-1}\left(\frac{N^d}{|A|}\right)\right)^p,$$

so

$$\|f\|_{L^p(\mu_A)} \le \left(\frac{|A|}{N^d}\right)^{\frac{1}{p}}\left(\Phi^{-1}\left(\frac{N^d}{|A|}\right)\right)\|f\|_{L^\Phi(\mu_A)},$$

as desired. □

Corollary 3.6. *Given a nice Young function Φ and $1 \le p < \infty$ such that $x^p \prec \Phi$, we have*

$$\|f\|_{L^p(\mu)} \le \left(\Phi^{-1}(1)\right)^p \|f\|_{L^\Phi(\mu)}.$$

Proof. Apply Lemma 3.5 on $A = \mathbb{Z}_N^d$. □

While the inequality from Theorem 3.3 has been useful, as evidenced by the results above, a more general version of Hölder's inequality will be required for our result. In particular, recall from the proof of Theorem 2.8 the use of the following trick

$$\|f\|_{L^p(A)} = \left(\sum_{x\in A}|f(x)|^p\right)^{\frac{1}{p}} = \left(\sum_{x\in A}|f^p(x)|\right)^{\frac{1}{p}} = \left(\|f^p\|_{L^1(A)}\right)^{\frac{1}{p}},$$

which allowed us to apply Hölder's inequality to any L^p norm, by first bringing it to L^1. Unfortunately, such a cheap trick will no longer work in Orlicz space; luckily, an analogous result exists which will allow us to apply Hölder's inequality to any L^Φ norm. Toward that result, we introduce the following lemmas.

Lemma 3.7 (Generalized Young's Inequality)**.** *If Φ, Ψ, Θ are Young functions satisfying*

$$\Phi^{-1}(x) \ge \Psi^{-1}(x)\Theta^{-1}(x)$$

for all $x \in [0,\infty)$, then for all $x, y \in [0,\infty)$,

$$\Phi(xy) \le \Psi(x) + \Theta(y).$$

Proof. It follows from the definition of Ψ^{-1} that $\Psi(\Psi^{-1}(x)) \le x \le \Psi^{-1}(\Psi(x))$. Take some $x, y \in [0,\infty)$, and suppose that $\Psi(x) \le \Theta(y)$. Then

$$\begin{aligned}\Phi(xy) &\le \Phi(\Psi^{-1}(\Psi(x))\Theta^{-1}(\Theta(y)))\\ &\le \Phi(\Psi^{-1}(\Theta(y))\Theta^{-1}(\Theta(y)))\\ &\le \Phi(\Phi^{-1}(\Theta(y)))\\ &\le \Theta(y).\end{aligned}$$

Similarly, if $\Psi(x) \geq \Theta(y)$, $\Phi(xy) \leq \Psi(x)$. As such,

$$\Phi(xy) \leq \max\{\Psi(x), \Theta(y)\} \leq \Psi(x) + \Theta(y),$$

as desired. □

Corollary 3.8. *If Φ, Ψ, Θ are Young functions satisfying*

$$\Phi^{-1}(x) \geq \Psi^{-1}(x)\Theta^{-1}(x)$$

for all $x \in [0, \infty)$, then for $f \in L^\Psi$, $g \in L^\Theta$, and $A \subseteq \mathbb{Z}_N^d$,

$$\sum_{x \in A} \Phi(|f(x)g(x)|) \leq \sum_{x \in A} \Psi(|f(x)|) + \sum_{x \in A} \theta(|g(x)|).$$

Proof. We sum over the inequality

$$\Phi(|f(x)g(x)|) \leq \Psi(|f(x)|) + \Theta(|g(x)|),$$

which follows immediately from Lemma 3.7. □

Lemma 3.9. *If Φ is a Young function, for $c \in (0, 1)$, $x \in [0, \infty)$,*

$$\Phi(cx) \leq c\Phi(x).$$

Proof. Take $\epsilon > 0$. By convexity,

$$\Phi(cx + (1 - c)\epsilon) \leq c\Phi(x) + (1 - c)\Phi(\epsilon).$$

Taking $\epsilon \to 0$, we obtain the desired result. □

Theorem 3.10 (Generalized Hölder's Inequality [3])**.** *If Φ, Ψ, Θ are Young functions satisfying*

$$\Phi^{-1}(x) \geq \Psi^{-1}(x)\Theta^{-1}(x)$$

for all $x \in [0, \infty)$, then for $f \in L^\Psi$, $g \in L^\Theta$, and $A \subseteq \mathbb{Z}_N^d$, $fg \in L^\Phi$ and

$$\|fg\|_{L^\Phi(A)} \leq 2\|f\|_{L^\Psi(A)}\|g\|_{L^\Theta(A)}.$$

Proof. Take some $u > \|f\|_{L^\Psi(A)}$, $v > \|g\|_{L^\Theta(A)}$. By Lemma 3.9 and Corollary 3.8,

$$\begin{aligned}\sum_{x \in A} \Phi\left(\frac{|f(x)g(x)|}{2uv}\right) &\leq \frac{1}{2}\sum_{x \in A} \Phi\left(\frac{|f(x)g(x)|}{uv}\right) \\ &\leq \frac{1}{2}\left(\sum_{x \in A} \Psi\left(\frac{|f(x)|}{u}\right) + \sum_{x \in A} \Psi\left(\frac{|g(x)|}{v}\right)\right) \\ &\leq \frac{1}{2}(1 + 1) = 1.\end{aligned}$$

Now let $u \to \|f\|_{L^\Psi(A)}$, $v \to \|g\|_{L^\Theta(A)}$. Then by definition of the Luxemburg norm,

$$\|fg\|_{L^\Phi(A)} \leq 2uv = 2\|f\|_{L^\Psi(A)}\|g\|_{L^\Theta(A)},$$

as desired. □

Finally, we are ready to introduce our main result.

4. Main Results

Definition 4.1. For nice complementary Young functions (Φ, Ψ), $x \prec \Phi \prec \Psi$, we say that $S \subset \mathbb{Z}_N^d$ satisfies a (Φ, Ψ) **Fourier restriction estimate** with uniform constant $C(\Phi, \Psi)$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\|\hat{f}\|_{L^\Psi(\mu_S)} \leq C(\Phi, \Psi) N^{-\frac{d}{2}} \|f\|_{L^\Phi(\mathbb{Z}_N^d)}.$$

Theorem 4.2. *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$, $E, S \subset \mathbb{Z}_N^d$, and suppose that a (Φ, Ψ) Fourier restriction estimate holds for S with constant $C(\Phi, \Psi)$. If $x \prec \Phi \prec x^2 \prec \Psi$ and*

$$|S| \left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^{-2} |E|^{-1} \leq \frac{N^d}{4C^2(\Phi, \Psi)(\Psi^{-1}(1))^4},$$

then

$$\|f\|_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{1}{1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2 |E| N^d}}}\right) \left(\|f\|_{L^2(E^\complement)} + \|\hat{f}\|_{L^2(S^\complement)}\right).$$

Proof. By the restriction assumption and Lemma 3.6,

$$\begin{aligned}
\|\widehat{1_E f}\|_{L^2(S)} &= |S|^{\frac{1}{2}} \|\widehat{1_E f}\|_{L^2(\mu_S)} \\
&\leq (\Psi^{-1}(1))^2 |S|^{\frac{1}{2}} \|\widehat{1_E f}\|_{L^\Psi(\mu_S)} \\
&\leq C(\Phi, \Psi)(\Psi^{-1}(1))^2 |S|^{\frac{1}{2}} N^{-\frac{d}{2}} \|f\|_{L^\Phi(E)}.
\end{aligned}$$

Let Θ be a Young function with $\Theta^{-1}(x) = x^{-\frac{1}{2}}\Phi^{-1}(x)$. Then by Theorem 3.10, the above quantity is bounded by

$$2C(\Phi, \Psi)(\Psi^{-1}(1))^2 |S|^{\frac{1}{2}} N^{-\frac{d}{2}} \|1\|_{L^\Theta(E)} \|f\|_{L^2(E)}.$$

Note that we may directly compute $\|1\|_{L^\Theta(E)}$ as

$$\begin{aligned}
|E|\Theta\left(\frac{1}{\|1\|_{L^\Theta(E)}}\right) &= 1 \\
\frac{1}{\|1\|_{L^\Theta(E)}} &= \Theta^{-1}\left(\frac{1}{|E|}\right) \\
\frac{1}{\|1\|_{L^\Theta(E)}} &= \Phi^{-1}\left(\frac{1}{|E|}\right) |E|^{\frac{1}{2}} \\
\|1\|_{L^\Theta(E)} &= \left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^{-1} |E|^{-\frac{1}{2}},
\end{aligned}$$

so that the above quantity can be rewritten as

$$\begin{aligned}
&2C(\Phi, \Psi)(\Psi^{-1}(1))^2 |S|^{\frac{1}{2}} N^{-\frac{d}{2}} \left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^{-1} |E|^{-\frac{1}{2}} \|f\|_{L^2(E)} \\
&= \sqrt{\frac{4C^2(\Phi, \Psi)(\Psi^{-1}(1))^4 |S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2 |E| N^d}} \|f\|_{L^2(E)}.
\end{aligned}$$

Now, via triangle inequality and Theorem 2.3,

$$\|\widehat{1_E f}\|_{L^2(S^\complement)} \geq \|\widehat{1_E f}\|_{L^2(\mathbb{Z}_N^d)} - \|\widehat{1_E f}\|_{L^2(S)}$$

$$\geq \|f\|_{L^2(E)} \left(1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2 |E|N^d}}\right),$$

hence

$$\begin{aligned}
\|f\|_{L^2(\mathbb{Z}_N^d)} &\leq \|f\|_{L^2(E)} + \|f\|_{L^2(E^\complement)} \\
&\leq \frac{\|\widehat{1_E f}\|_{L^2(S^\complement)}}{1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2|E|N^d}}} + \|f\|_{L^2(E^\complement)} \\
&= \frac{\|\hat{f} - \widehat{1_{E^\complement} f}\|_{L^2(S^\complement)}}{1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2|E|N^d}}} + \|f\|_{L^2(E^\complement)} \\
&= \frac{\|\hat{f}\|_{L^2(S^\complement)} + \|f\|_{L^2(E^\complement)}}{1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2|E|N^d}}} + \|f\|_{L^2(E^\complement)} \\
&\leq \left(1 + \frac{1}{1 - \sqrt{\frac{4C^2(\Phi,\Psi)(\Psi^{-1}(1))^4|S|}{\left(\Phi^{-1}\left(\frac{1}{|E|}\right)\right)^2|E|N^d}}}\right)\left(\|f\|_{L^2(E^\complement)} + \|\hat{f}\|_{L^2(S^\complement)}\right).
\end{aligned}$$

□

References

[1] Saifallah Ghobber and Philippe Jaming. On uncertainty principles in the finite dimensional setting. *Linear Algebra and its Applications*, 435(4):751–768, August 2011. 3

[2] Philippe Jaming, Alexander Iosevich, and Azita Mayeli. Uncertainty Principle, annihilating pairs and Fourier restriction. working paper or preprint, February 2025. 1, 5

[3] Richard O'Neil. Fractional integration in orlicz spaces. i. *Transactions of the American Mathematical Society*, 115:300–328, 1965. 11

[4] M. M. (Malempati Madhusudana) Rao and Z. D. Ren. *Theory of Orlicz spaces / M.M. Rao, Z.D. Ren.* Monographs and textbooks in pure and applied mathematics ; 146. M. Dekker, New York, 1991. 8