A Number of Perspectives on Signal Recovery

William Hagerstrom

Abstract

We discuss signal recovery in three settings: $\mathbb{Z}_N^d$, $\mathbb{R}^d$, and the $SU(2)$ nonlinear Fourier transform series. In particular, we explore the relationship between uncertainty principles, unique signal recovery, and restriction theory. In $\mathbb{Z}_N^d$ and $\mathbb{R}^d$, we provide mechanisms for recovery.

1 Introduction

Heuristically, the uncertainty principle in harmonic analysis is the notion that a function and its Fourier transform cannot both be "simple." While this concept can take different rigorous forms (see [5]), one version in the setting of $\mathbb{Z}_N^d$ is that $|\text{supp}(f)||\text{supp}(\hat{f})| \geq N^d$. The study of the connection between uncertainty principles of this type and signal recovery was elucidated in [2].

Restriction theory is the study of sets $S \subset G$ for which an inequality of the form

$$||\hat{f}||_{L^q(S)} \lesssim ||f||_{L^p(G)} \tag{1}$$

holds. For example, the Hausdorff-Young inequality states that $S = G = \mathbb{R}^d$ satisfies (1) for $p < 2$ and $q = p'$. The authors of [6] introduced restriction theory to the problem of signal recovery by showing that improvements can be made to uncertainty principles when $\hat{f}$ is supported in a set satisfying a nontrivial restriction estimate. This connection has been further developed in [4] and [7].

In Sections 2 and 3, we provide an overview of the results that can be obtained by these concepts in the setting of $\mathbb{Z}_N^d$ and $\mathbb{R}^d$. In Section 4, we introduce the $SU(2)$ nonlinear Fourier series (NLFS) from [1]. While there are technical difficulties in translating arguments to the nonlinear setting, we use signal recovery on S^1 to prove a unique recovery result for the NLFS.

Remark. Some readers may be interested in Appendix 6.4, which covers an extension of Carleson's theorem to $\mathbb{R}^d$. While some sources ([3] and [8]) allude to the result proven here, the author was unable to find a satisfactory proof in the literature.

2 Signal Recovery in $\mathbb{Z}_N^d$

2.1 Summary of the Fourier Transform in $\mathbb{Z}_N^d$

Throughout this section, a signal will refer to a function $f : \mathbb{Z}_N^d \to \mathbb{C}$. We use $\chi : \mathbb{Z}_N \to \mathbb{C}$ to denote the character $\chi(a) = e^{-2\pi i a/N}$. Copying conventions from Euclidean space, we define $x \cdot y = \sum_{i=1}^d x_i y_i$ for $x, y \in \mathbb{Z}_N^d$. We are now ready to introduce the Fourier transform.

Definition *For a signal f, we define $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$, the Fourier transform of f, by*

$$\hat{f}(m) = N^{-d/2} \sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x).$$

Remark. It should be noted that the choice of normalizing constant in the definition above varies throughout the literature. We choose $N^{-d/2}$ so that the Plancherel theorem has constant 1.

An essential result for the Fourier transform is the inversion theorem.

Theorem 1 (Fourier Inversion) *For a signal f, we have*

$$f(x) = N^{-d/2} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m).$$

PROOF: Plugging in the definition of $\hat{f}$ gives us

$$N^{-d/2} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m) = N^{-d} \sum_{m \in \mathbb{Z}_N^d} \sum_{y \in \mathbb{Z}_N^d} \chi(-y \cdot m) \chi(x \cdot m) f(y).$$

Interchanging the order of summation and using properties of of the exponential, the above is equal to

$$N^{-d} \sum_{y \in \mathbb{Z}_N^d} f(y) \sum_{m \in \mathbb{Z}_n^d} \chi(m \cdot (x - y)).$$

As shown in Appendix 6.1, the inner sum is nonzero only if $x = y$. In this case it is N^d. Hence, the sums collapse to $f(x)$. This is the desired result. □

Remark. If we let $\{X_1, \dots, X_{N^d}\}$ be some enumeration of $\mathbb{Z}_N^d$ and view a function f as the vector $(f(X_1), \dots, f(X_{N^d}))$, then the operator mapping f to $\hat{f}$ has the matrix representation $\hat{f} = Af$, where $A_{ij} = N^{-d/2} \chi(-X_i \cdot X_j)$, $1 \le i, j \le N^d$. In light of this, the proof above shows that A is an invertible matrix and $(A^{-1})_{ij} = N^{-d/2} \chi(X_i \cdot X_j)$. This perspective was explored systematically in [9].

As alluded to, we have the following result known as the Plancherel theorem.

Theorem 2 (Plancherel) *For a signal f, we have $||f||_{L^2(\mathbb{Z}_N^d)} = ||\hat{f}||_{L^2(\mathbb{Z}_N^2)}$.*

PROOF: We will work from the Fourier transform side and unwind definitions. We have

$$\begin{aligned} ||\hat{f}||^2_{L^2(\mathbb{Z}_N^d)} &= N^{-d} \sum_{m \in \mathbb{Z}_N^d} \left(\sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x) \right) \overline{\left(\sum_{y \in \mathbb{Z}_N^d} \chi(-y \cdot m) f(y) \right)} \\ &= N^{-d} \sum_{x,y \in \mathbb{Z}_N^d} f(x)\overline{f(y)} \sum_{m \in \mathbb{Z}_N^d} \chi(m \cdot (y - x)) \end{aligned}$$

The second line is achieved by the properties of the exponential and interchanging the order of summation. As in the proof of the inversion theorem, the inner sum is N^d when $y = x$ and 0 otherwise. So, the last line becomes $\sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 = ||f||^2_{L^2(\mathbb{Z}_N^d)}$. Taking square roots gives the desired result. □

2.2 Signal Recovery in $\mathbb{Z}_N^d$

With the necessary machinery now in place, we return to the question that defines the theme of this paper: What can be known about a signal f if some values of $\hat{f}$ are unobserved? For a general signal f, the invertibility of the Fourier transform shows that the loss of any Fourier coefficient $\hat{f}(m)$ will result in the loss of f. However, by placing restrictions f, we can obtain a positive result.

Definition *The support of a signal f is supp$(f) = \{x \in \mathbb{Z}_N^d : f(x) \neq 0\}$.*

For some signal f, let $E = \text{supp}(f)$ and $S = \text{supp}(\hat{f})$. To derive a recovery result, we start with the following computation:

$$\begin{aligned} |f(x)| &= N^{-d/2} \left| \sum_{m \in S} \chi(x \cdot m) \hat{f}(m) \right| \\ &\leq N^{-d/2} \sum_{m \in S} |\hat{f}(m)| \\ &\leq N^{-d} |S| ||f||_{L^1(\mathbb{Z}_N^d)}. \end{aligned}$$

In the last step, we bound the sum by the number of terms times the size of the largest term and the bound from Appendix 6.2 that $|\hat{f}(m)| \leq N^{-d/2} ||f||_{L^1(\mathbb{Z}_N^d)}$. Summing the inequality over $x \in E$ and noting the the RHS is independent of x, we get $||f||_{L^1(\mathbb{Z}_N^d)} \leq N^{-d}|S||E|||f||_{L^1(\mathbb{Z}_N^d)}$. If f is nonzero, we can divide by the 1-norm of f to get $|S||E| \geq N^d$. To summarize, we have proven the following.

Theorem 3 (Uncertainty Principle) *If f is a nonzero signal, then $|supp(f)||supp(\hat{f})| \geq N^d$.*

Suppose we have a signal f with a support of known size $|\text{supp}(f)|$. If $\hat{f}(m)$ is unobserved for $m \in S$, we will have unique recovery of f if there is no signal $g \neq f$ such that $\hat{g}(m) = \hat{f}(m)$ for $m \notin S$ and $|\text{supp}(g)| = |\text{supp}(f)|$. Suppose some function g of this type exists. Consider the function $h = f - g$. The support of h has size at most $2|\text{supp}(f)|$ and the support of $\hat{h}$ is contained in S. By Theorem 3, we have $2|\text{supp}(f)|||S| \geq N^d$. If we suppose that this inequality does not hold, then the existence of g supplies a contradiction. In summary, we have the following recovery result.

Theorem 4 (Unique Recovery) *If f is a signal and $\hat{f}(m)$ is unobserved for $m \in S$, then f can be uniquely recovered if $|supp(f)||S| < \frac{N^d}{2}$.*

2.3 Recovery Mechanisms in $\mathbb{Z}_N^d$

It remains to provide a mechanism by which we can recover f. In this section, we will explore two methods of recovery. The first is the Direct Rounding Algorithm (DRA) introduced in [6], which provides recovery in the case that f is the indicator function of some set. The second is Logan's method from [2], and is a computationally efficient way to recover general signals f. Both require that the bound of Theorem 4 is satisfied.

To state the DRA, we define a rounding function $R : \mathbb{C} \to \{0, 1\}$ by

$$R(x) = \begin{cases} 0 & |x| < \frac{1}{2} \\ 1 & |x| \geq \frac{1}{2} \end{cases}.$$

Theorem 5 (Direct Rounding Algorithm) *Let $E \subset \mathbb{Z}_N^d$. If $S \subset \mathbb{Z}_N^d$ and $|S||E| < \frac{N^d}{2}$, then*

$$1_E(x) = R\left(N^{-d/2} \sum_{m \notin S} \chi(x \cdot m) \widehat{1_E}(m) \right).$$

PROOF: By the Fourier inversion formula,

$$1_E(x) = N^{-d/2} \sum_{m \notin S} \chi(x \cdot m) \widehat{1_E}(m) + N^{-d/2} \sum_{m \in S} \chi(x \cdot m) \widehat{1_E}(m).$$

By the ∞-norm bound on $\widehat{1_E}$, the size of the second sum in bounded by $N^{-d}|S||E| < \frac{1}{2}$. So,

$$\left| 1_E(x) - N^{-d/2} \sum_{m \notin S} \chi(x \cdot m) \widehat{1_E}(m) \right| < \frac{1}{2}.$$

So, $1_E(x) = 0$ only if $N^{-d/2} \sum_{m \notin S} \chi(x \cdot m) \widehat{1_E}(m) < \frac{1}{2}$ and $1_E(x) = 1$ only if $N^{-d/2} \sum_{m \notin S} \chi(x \cdot m) \widehat{1_E}(m) > \frac{1}{2}$. Since 1_E takes on only 0 and 1 as values, this implies the result. □

To state Logan's method, we will make use of the following definition.

Definition *For a function $f : X \to \mathbb{R}_{\geq 0}$ and $A \subset X$, we define $\operatorname{argmin}_{x \in A} f(x)$ to be the set of points x^* such that $f(x^*) \leq f(x)$ for $x \in A$.*

Note that the $\operatorname{argmin}_{x \in A} f(x)$ could be an empty set. For example, $\operatorname{argmin}_{x \in (0,1)} x$ is empty. In Appendix 6.3, we prove that the argmin used in the the proofs of Theorem 6 and Theorem 9 is nonempty.

Theorem 6 (Logan's Method) *Let f be a signal supported in E and suppose $\hat{f}(m)$ is unobserved for some set $S \subset \mathbb{Z}_N^d$. If $\frac{|S||E|}{N^d} < \frac{1}{2}$, then f is the only function contained in $\operatorname{argmin}_{u \in A} ||u||_{L^1(\mathbb{Z}_N^d)}$. Here, A is the set of signals u such that*

$\hat{u}(m) = \hat{f}(m)$ *for* $m \notin S$.

PROOF: Let $g \in \text{argmin}_{u \in A} ||u||_{L^1(\mathbb{Z}_N^d)}$. Suppose $g \neq f$ and define $h = f - g$. Then,

$$\begin{aligned} ||g||_{L^1(\mathbb{Z}_N^d)} &= ||g||_{L^1(E)} + ||g||_{L^1(E^c)} \\ &= ||f - h||_{L^1(E)} + ||h||_{L^1(E^c)} \\ &\geq ||f||_{L^1(\mathbb{Z}_N^d)} + ||h||_{L^1(E^c)} - ||h||_{L^1(E)}. \end{aligned}$$

The second line follows from the fact that f is supported on E and the third follows from the triangle inequality. Now,

$$\begin{aligned} |h(x)| &= N^{-d/2} \left| \sum_{m \in S} \chi(x \cdot m) \hat{h}(m) \right| \\ &\leq N^{-d} |S| ||h||_{L^1(\mathbb{Z}_N^d)}. \end{aligned}$$

Summing over E, we get

$$\begin{aligned} ||h||_{L^1(E)} &\leq \frac{|S||E|}{N^d} ||h||_{L^1(\mathbb{Z}_N^d)} \\ &< \frac{||h||_{L^1(\mathbb{Z}_N^d)}}{2}. \end{aligned}$$

But, this implies that $||h||_{L^1(E^c)} - ||h||_{L^1(E)} > 0$. So, $||g||_{L^1(\mathbb{Z}_N^d)} > ||f||_{L^1(\mathbb{Z}_N^d)}$. This contradicts our definition of g. So, we must have $f = g$. □

2.4 Restriction in $\mathbb{Z}_N^d$

The object of study in this section is the following:

Definition *Let* $1 \leq p \leq q \leq \infty$. *A set* $S \subset \mathbb{Z}_N^d$ *satisfies a* (p, q)*-restriction estimate with constant* $C_{p,q}$ *if we have*

$$\left(\frac{1}{|S|} \sum_{m \in S} |\hat{f}|^q \right)^{1/q} \leq C_{p,q} N^{-d/2} ||f||_{L^p(\mathbb{Z}_N^d)}.$$

for all signals f.

We will see that in the presence of a restriction estimate on a set S, we can improve on the results from the previous section. Suppose we have a $(p, 2)$-restriction on some set S with constant C. Let f be a nonzero signal with support in E and Fourier support in S. Plancherel's theorem, the restriction estimate, and Holder's inequality give us

$$\begin{aligned}||f||_{L^2(\mathbb{Z}_N^d)} &= ||\hat{f}||_{L^2(S)} \\ &\leq C|S|^{1/2}N^{-d/2}||f||_{L^p(\mathbb{Z}_N^d)} \\ &\leq C|S|^{1/2}N^{-d/2}||f||_{L^2(\mathbb{Z}_N^d)}^{2-(2/p)}||f||_{L^1(\mathbb{Z}_N^d)}^{(2/p)-1}.\end{aligned}$$

So,

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq C'|S|^{p/(4-2p)}N^{-dp/(4-2p)}||f||_{L^1(\mathbb{Z}_N^d)}. \tag{2}$$

Applying Holder's inequality again and dividing by the 2-norm of f, we get

$$1 \leq C'|S|^{p/(4-2p)}N^{-dp/(4-2p)}|E|^{1/2}.$$

To summarize, we have the following result:

Theorem 7 (Uncertainty Principle with Restriction) *Suppose $S \subset \mathbb{Z}_N^d$ satisfies a $(p, 2)$-restriction estimate with constant C. If f is a nonzero signal with support E and Fourier support in S, then there is some constant C' depending on p and C such that*

$$|E|^{(2/p)-1}|S| \geq C'N^d.$$

In a similar manner to the last section, we will convert Theorem 7 into a unique recovery result.

Theorem 8 *Suppose $S \subset \mathbb{Z}_N^d$ satisfies a $(p, 2)$-restriction estimate. Let C' be as in Theorem 7. Let f be a signal supported in E. If $\hat{f}(m)$ is unobserved for $m \in S$ and $|E|^{(2/p)-1}|S| < \frac{C'N^d}{2^{(2/p)-1}}$, then f can be uniquely recovered.*

PROOF: Suppose g is a signal with support F such that $|F| = |E|$ and $\hat{g}(m) = \hat{f}(m)$ for $m \notin S$. If $f \neq g$, then $h = f - g$ is a nonzero signal with support of size at most $2|E|$ and Fourier support contained in S. By Theorem 7, we have

$$2^{(2/p)-1}|E|^{(2/p)-1}|S| \geq C'N^d.$$

After dividing by $2^{(2/p)-1}$, we see that this contradicts our assumption. Thus, we must have $f = g$. □

We can also improve Theorem 6 in the presence of restriction.

Theorem 9 (Logan's Method with Restriction) *Suppose $S \subset \mathbb{Z}_N^d$ satisfies a $(p, 2)$-restriction estimate. Let C' be as in Theorem 7. Let f be a signal and suppose $\hat{f}(m)$ is unobserved for $m \in S$. If $C'|E|^{1/2}|S|^{p/(4-2p)}N^{-dp/(4-2p)} < \frac{1}{2}$,* *then f is the only function contained in* $\text{argmin}_{u\in A}||u||_{L^1(\mathbb{Z}_N^d)}$. *Here, A is the set of signals u such that $\hat{u}(m) = \hat{f}(m)$ for $m \notin S$.*

PROOF: As before, let $g = \text{argmin}_{u\in A}||u||_{L^1(\mathbb{Z}_N^d)}$, assume that $f \neq g$, and let $h = f - g$ (Again, the existence of g follows from Appendix 6.3). Following the same steps as in the proof of Theorem 6, we have

$$||g||_{L^1(\mathbb{Z}_N^d)} \geq ||f||_{L^1(\mathbb{Z}_N^d)} + ||h||_{L^1(E^c)} - ||h||_{L^1(E)}.$$

Instead of bounding h pointwise, we use Cauchy-Schwarz and (2) to get

$$\begin{aligned} ||h||_{L^1(E)} &\leq |E|^{1/2}||h||_{L^2(\mathbb{Z}_N^d)} \\ &\leq C'|S|^{p/(4-2p)}N^{-dp/(4-2p)}||h||_{L^1(\mathbb{Z}_N^d)}. \end{aligned}$$

By assumption, this quantity is strictly bounded by $\frac{1}{2}||h||_{L^1(\mathbb{Z}_N^d)}$. Thus, $||h||_{L^1(E^c)} - ||h||_{L^1(E)} > 0$. So, we get the contradiction $||f||_{L^1(\mathbb{Z}_N^d)} < ||g||_{L^1(\mathbb{Z}_N^d)}$. Thus, we must have $f = g$. □

3 Signal Recovery in $\mathbb{R}^d$

3.1 Summary of The Fourier Transform in $\mathbb{R}^d$

As the development of the Fourier transform on $\mathbb{R}^d$ is lengthy and technical, we will not prove the main theorems. For a reference on the topic, see [10]. We now outline the results needed for our purposes.

Definition *For a finite measure μ of bounded variation, we define $\hat{\mu} : \mathbb{R}^d \to \mathbb{C}$, the Fourier transform of μ, to be*

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}d\mu.$$

Theorem 10 (Fourier Inversion) *If $f \in L^1(\mathbb{R}^d)$ is a complex valued function and $\hat{f} \in L^1(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi.$$

The assumption that $\hat{f} \in L^1(\mathbb{R}^d)$ will be too restrictive for our purposes. In Appendix 6.4, we will adapt methods from [3] and [8] to prove the following:

Theorem 11 (Carleson's Theorem in $\mathbb{R}^d$) *Let $P \subset \mathbb{R}^d$ be the convex hull of the points in $\mathbb{R}^d$ with each coordinate being* ± 1. *If $f \in L^2(\mathbb{R}^d)$, then for almost every x we have*

$$f(x) = \lim_{r\to\infty}\int_{rP} e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi.$$

3.2 Restriction in $\mathbb{R}^d$

To begin our discussion of restriction on $\mathbb{R}^d$, we give the following definition:

Definition *For a set $S \subset \mathbb{R}^d$ and a finite measure μ on S, we say that we have a (p,q)-restriction estimate on S relative to the measure μ if for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have $||\hat{f}||_{L^q(d\mu)} \leq C||f||_{L^p(\mathbb{R}^d)}$.*

It is conjectured that a (p,q)-restriction holds on S^{d-1} with respect to the surface measure whenever $p < \frac{2d}{d+1}$ and $q \leq \frac{d-1}{d+1}p'$. The celebrated Stein-Tomas theorem is a partial result in this direction.

Theorem 12 (Stein-Tomas) *A $(p,2)$-restriction estimate on S^{d-1} with respect to σ, the surface measure, holds when $p \leq \frac{2d+2}{d+3}$.*

For a proof of the Stein-Tomas theorem, see [10]. We are interested in recovering f if $\hat{f}$ is lost on some set of positive Lebesgue measure. So, we will derive a version of the Stein-Tomas theorem on a "thickened" sphere. Let

$$S^\delta = \{x \in \mathbb{R}^d : 1 - \delta/2 < |x| < 1 + \delta/2\}.$$

and μ be the restriction of the Lebesgue measure to S^δ. For $p \leq \frac{2d+2}{d+3}$, we have

$$\begin{aligned}
\int_{S^\delta} |\hat{f}(\xi)|^2 d\xi &= \int_{1-\delta/2}^{1+\delta/2} r^{d-1} \int_{S^{d-1}} |\hat{f}(r\omega)|^2 d\omega dr \\
&= \int_{1-\delta/2}^{1+\delta/2} r^{-1-d} \int_{S^{d-1}} |\widehat{f(\cdot/r)}(\omega)|^2 d\omega dr \\
&\leq C_p^2 \int_{1-\delta/2}^{1+\delta/2} r^{-1-d} ||f(\cdot/r)||^2_{L^p(\mathbb{R}^n)} dr \\
&= C_p^2 ||f||^2_{L^p(\mathbb{R}^n)} \int_{1-\delta/2}^{1+\delta/2} r^{-1-d+\frac{2d}{p}} dr.
\end{aligned}$$

Assuming that $0 < \delta < 1$, the integrand is bounded above and below by a constant D depending only on p and d. So, the integral is bounded by $D\delta$. Consolidating our constants and taking square roots, we get

$$||\hat{f}||_{L^2(d\mu)} \leq C'_p \delta^{1/2} ||f||_{L^p(\mathbb{R}^d)}.$$

To summarize, we have the following:

Theorem 13 (Stein-Tomas on a Thick Sphere) *A $(p,2)$-restriction theorem holds on $S^\delta \subset \mathbb{R}^d$ with respect to the restriction of the Lebesgue measure with constant $C_p\delta^{1/2}$ whenever $p \leq \frac{2d+2}{d+3}$.*

3.3 Signal Recovery in $\mathbb{R}^d$

We are now equipped for our discussion of signal recovery in $\mathbb{R}^d$. As promised, we return to the Direct Rounding Algorithm from [6]. Suppose $A \subset \mathbb{R}^n$ set with finite positive Lebesgue measure. By Theorem 11, for almost every x

we have

$$1_A(x) = \lim_{r\to\infty} \int_{rP} e^{2\pi i \xi\cdot x}\widehat{1_A}(\xi)d\xi + \int_{S^\delta} e^{2\pi i\xi\cdot x}\widehat{1_A}(\xi)d\xi.$$

Call the integrals (or limits of integrals) above I and II. As in the $\mathbb{Z}_N^d$ case, we will be interested in bounding II. Applying Holder's inequality gives

$$II \le |S^\delta|^{1/2}||\widehat{1_A}||_{L^2(d\mu)}.$$

Stein-Tomas bounds this further by

$$C\delta^{1/2}|S^\delta|^{1/2}||1_A||_{L^{(2d+2)/(d+3)}(\mathbb{R}^d)} = C\delta^{1/2}|S^\delta|^{1/2}|A|^{(d+3)/(2d+2)}.$$

Up to a constant depending on d, $S^\delta = \delta$. So,

$$\left|\int_{S^\delta} e^{2\pi i\xi\cdot x}\widehat{1_A}(\xi)d\xi\right| \le C'\delta|A|^{(d+3)/(2d+2)}.$$

If $|A| < \left(C'\delta\right)^{-(2d+2)/(d+3)}$, then $|II| < \frac{1}{2}$ and the DRA recovers A away from a set of measure 0. This computation is performed in [6] under the assumption that the restriction conjecture holds. In that case, the exponent improves to $-\frac{2d}{d+1}$.

Now, we will be less restrictive. Instead of requiring that the DRA recovers A away from a set of measure 0, lets bound the size of the error set. Define

$$f(x) = \int_{S^\delta} e^{2\pi i\xi\cdot x}\widehat{1_A}(\xi)d\xi \qquad\qquad B = \left\{x : |f(x)| \ge \frac{1}{2}\right\}.$$

We clearly get $||f||_{L^2(\mathbb{R}^d)} \ge \frac{\sqrt{|B|}}{2}$. An alternative way to write f is $(\widehat{1_A}1_{S^\delta})^\vee$. From this perspective, the Plancherel theorem and the Stein-Tomas theorem give

$$\begin{aligned} ||f||_{L^2(\mathbb{R}^d)} &= ||\widehat{1_A}||_{L^2(d\mu)} \\ &\le C\sqrt{\delta}||1_A||_{L^{(2d+2)/(d+3)}(\mathbb{R}^d)} \\ &= C\sqrt{\delta}|A|^{(2d+2)/(d+3)}. \end{aligned}$$

So, $|B| \le C'\delta|A|^{(4d+4)/(d+3)}$. In summary, we have the following results:

Theorem 14 (Exact DRA) *Suppose $A \subset \mathbb{R}^d$ is a set with finite positive Lebesgue measure. If $\widehat{1_A}$ is unobserved outside of S^δ and $|A| < (C\delta)^{-(2d+2)/(d+3)}$, then A can be recovered by the DRA. Here, C depends only on d.*

Theorem 15 (Approximate DRA) *Suppose $A \subset \mathbb{R}^d$ is a set with finite positive Lebesgue measure. If $\widehat{1_A}$ is unobserved* *outside of S^δ then the error set of the DRA does not have Lebesgue measure greater than $C'\delta|A|^{(4d+4)/(d+3)}$. Here, C depends only on d.*

4 Signal Recovery for the Nonlinear Fourier Series

4.1 Introduction to Nonlinear Fourier Series

We begin by building up the necessary machinery in nonlinear Fourier analysis. We loosely follow [1]. For a set $D \subset \mathbb{C}$, we define the reflected set $D^* = \{\overline{z^{-1}} : z \in D\}$. We define the reflection of a function $a : D \subset \mathbb{C} \to \mathbb{C}$ by $a^*(z) = \overline{a\left(\overline{z^{-1}}\right)}$. As defined, if a is meromorphic on D, then a^* is meromorphic on D^*.

Let $\{F_n\}_{n\in\mathbb{Z}}$ be a finitely supported sequence in $\mathbb{Z}$ with values in $\mathbb{C}$. Define the $SU(2)$ nonlinear Fourier transform of $\{F_n\}$ by

$$\tilde{F}(z) = \prod_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+|F_n|^2}} \begin{pmatrix} 1 & F_n z^n \\ -\overline{F_n} z^{-n} & 1 \end{pmatrix}.$$

We interpret products of this form from left to right in increasing index. Note that the infinite product collapses to a finite one given that factors outside the support of $\{F_n\}$ are the identity.

Lemma 1 *The nonlinear Fourier transform of $\{F_n\}$, a complex-valued sequence with finite support, is of the form*

$$\tilde{F}(z) = \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix}.$$

for meromorphic functions a, b such that $aa^* + bb^* = 1$ on the unit disk.

PROOF: Let m be the least value for which F_m is nonzero and M be the greatest value for which F_M is nonzero. We have that

$$\tilde{F}(z) = \prod_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+|F_n|^2}} \begin{pmatrix} 1 & F_n z^n \\ -\overline{F_n} z^{-n} & 1 \end{pmatrix} = \prod_{n=m}^{M} A_n$$

where

$$A_n = \begin{pmatrix} (1+|F_n|^2)^{-1/2} & (1+|F_n|^2)^{-1/2} F_n z^n \\ -(1+|F_n|^2)^{-1/2} \overline{F_n} z^{-n} & (1+|F_n|^2)^{-1/2} \end{pmatrix}.$$

It is easy to see that each A_n is of the desired form. By induction, it suffices to show that the product of two matrix functions satisfying the properties stated in the theorem satisfies these same properties. Let

$$P(z) = \begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} \qquad\qquad Q(z) = \begin{pmatrix} c(z) & d(z) \\ -d^*(z) & c^*(z) \end{pmatrix}$$

be two such matrix functions. Then,

$$\begin{aligned}(PQ)(z) &= \begin{pmatrix} (ac - bd^*)(z) & (ad + bc^*)(z) \\ (-cb^* - a^*d^*)(z) & (-b^*d + a^*c^*)(z) \end{pmatrix} \\ &= \begin{pmatrix} (ac - bd^*)(z) & (ad + bc^*)(z) \\ -(ad + bc^*)^*(z) & (ac - bd^*)^*(z) \end{pmatrix}.\end{aligned}$$

We can now read off that the matrix function PQ has the desired form and that its entries are meromorphic. The fact that its determinant is 1 follows from the homomorphism property of the determinant. This completes the proof. □

From this point on, we understand the row vector (a, b) as the matrix $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$. Thus, multiplication of row vectors is given by

$$(a, b)(c, d) = (ac - bd^*, ad + bc^*).$$

Now, we will derive the formulas for a and b in terms of $\{F_n\}$ that will provide the means for signal recovery. We begin by decomposing the definition of the nonlinear Fourier transform using the fact that $(1, F_n z^n) = (1, 0) + (0, F_n z^n)$.

$$\begin{aligned}(a, b) &= \prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2}(1, F_n z^n) \\ &= \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \left(\prod_{n=-\infty}^{\infty} ((1, 0) + (0, F_n z^n)) \right) \\ &= \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \left(\sum_{k=0}^{\infty} \sum_{j_1 < \cdots j_k} \prod_{n=1}^{k} (0, F_{j_n} z^{j_n}) \right).\end{aligned}$$

Here, the empty $k = 0$ term is to be treated as the identity matrix. Note that $\prod_{n=1}^{k}(0, F_n z^n)$ is antidiagonal for odd k and diagonal for even k. In either case, the nonzero entry in the row vector will be

$$\left(\prod_{\substack{1 \le n \le k \\ n \text{ odd}}} F_{j_n} z^{j_n} \right) \left(\prod_{\substack{1 \le n \le k \\ n \text{ even}}} -\overline{F_{j_n}} z^{-j_n} \right).$$

Thus,

$$a(z) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{j_1 < \cdots j_{2k}} \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ odd}}} F_{j_n} z^{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ even}}} -\overline{F_{j_n}} z^{-j_n} \right). \tag{3}$$

and

$$b(z) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{j_1 < \cdots j_{2k+1}} \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ odd}}} F_{j_n} z^{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} z^{-j_n} \right). \tag{4}$$

Given our treatment of the $k = 0$ term above, the $k = 0$ term in the formula for a is 1. Notice that for a fixed $k > 0$ and $j_1 < \cdots j_{2k}$, we have that the power of z in the product (3) is

$$\sum_{n=1}^{2k} (-1)^{k+1} j_k < 0.$$

Since $\{F_n\}$ is finitely supported, we can freely interchange sums and integrals to get

$$\frac{1}{2\pi} \int_{S^1} a = \prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2}. \tag{5}$$

We can generalize this computation to derive formulas for the Fourier coefficients of a and b in terms of the sequence $\{F_n\}$. For a fixed $p \in \mathbb{Z}$, letting $z = e^{it}$ and multiplying (3) by e^{-ipt} gives

$$a(e^{it}) e^{-ipt} = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{j_1 < \cdots j_{2k}} e^{-ipt} \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ odd}}} F_{j_n} z^{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ even}}} -\overline{F_{j_n}} z^{-j_n} \right).$$

Integrating in t from 0 to 2π gives

$$\hat{a}(p) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots j_{2k} \\ \sum_{n=1}^{2k} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ even}}} -\overline{F_{j_n}} \right). \tag{6}$$

We similarly get for b that

$$\hat{b}(p) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k+1} \\ \sum_{n=1}^{2k+1} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} \right). \tag{7}$$

4.2 Signal Recovery for the Nonlinear Fourier Series

In this section, we will utilize (7) and results from [2] to establish the following recovery result for the nonlinear Fourier transform.

Theorem 16 *Suppose $\{F_n\}$ is a complex valued sequence with finite support $W \subset \mathbb{Z}$ and let $(a, b) = \tilde{F}$. If b is known on S^1 outside of some measurable set and*

$$|E| \min \left\{ diam(W) + 1, 2^{|W|-1} \right\} < \frac{1}{2},$$

then b can be exactly recovered on S^1.

The necessary result is an adaptation of Theorem 4 in [2]. In the source, it is stated in terms of the continuous linear Fourier transform and a signal with noise. In Appendix 6.5, we prove the following:

Theorem 17 *Suppose $\hat{f}$ is supported on a finite set $W \subset \mathbb{Z}$ and f is known outside of a set $E \subset S^1$. If $|W||E| < \frac{1}{2}$, then f can be reconstructed exactly. Here S^1 is given the surface measure so that $|S^1| = 1$.*

It is clear by comparing this result and the result stated at the beginning of this section that it suffices to prove the following:

Lemma 2 *If $\{F_n\}$ is a complex valued sequence with finite support $W \subset \mathbb{Z}$ and $(a, b) = \tilde{F}$, then*

$$|supp(\hat{b})| \leq \min\left\{diam(W) + 1, 2^{|W|-1}\right\}.$$

PROOF: We begin with (7):

$$\hat{b}(p) = \left(\prod_{n=-\infty}^{\infty}(1+|F_n|^2)^{-1/2}\right)\sum_{k=0}^{\infty}\sum_{\substack{j_1<\cdots<j_{2k+1}\\ \sum_{n=1}^{2k+1}(-1)^{n+1}j_n=p}}\left(\prod_{\substack{1\leq n\leq 2k+1\\ n\text{ odd}}}F_{j_n}\right)\left(\prod_{\substack{1\leq n\leq 2k+1\\ n\text{ even}}}-\overline{F_{j_n}}\right).$$

By the formula above, we have that

$$\text{supp}(\hat{b}) \subset \left\{p \in \mathbb{Z} : \text{there exists } k \in \mathbb{Z}_{\geq 0} \text{ and } j_1 < \cdots < j_{2k+1} \in W \text{ such that } \sum_{n=1}^{2k+1}(-1)^{n+1}j_n = p\right\}. \tag{8}$$

Let M and m be the sharpest upper and lower bounds on the support of $\{F_n\}$. Let $j_1 < \cdots < j_{2k+1} \in W$. Then,

$$\begin{aligned}\sum_{n=1}^{2k+1}(-1)^{n+1}j_n &= j_{2k+1} + \sum_{n=1}^{2k}(-1)^{n+1}j_n\\ &= j_{2k+1} + \sum_{n=1}^{k} j_{2n-1} - j_{2n}\\ &\leq M.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{2k+1}(-1)^{n+1}j_n &= j_1 + \sum_{n=2}^{2k+1}(-1)^{n+1}j_n\\ &= j_1 + \sum_{n=1}^{k} j_{2n+1} - j_{2n}\\ &\geq m.\end{aligned}$$

This proves that $\text{supp}(\hat{b}) \subset [m, M]$. So, $|\text{supp}(\hat{b})| \leq \text{diam}(W) + 1$.

It remains to show that $|\text{supp}(\hat{b})| \leq 2^{|W|}$. The worst case is that every choice of $j_1 < \cdots < j_{2k+1}$ gives a different value for

$$\sum_{n=1}^{2k+1} (-1)^{n+1} j_n.$$

In this worst case, $|\text{supp}(\hat{b})|$ is bounded by the number of odd sized subsets of W. This is equal to $2^{|W|-1}$. □

Remark. This bound is sharp in the sense that we can construct sequences $\{F_n\}$ with arbitrarily large support such that $\text{supp}(\hat{b}) = \min\left(\text{diam}(W) + 1, 2^{|W|-1}\right)$. Further, we can construct such sets when the minimum is $\text{diam}(W) + 1$ and when the minimum is $2^{|W|-1}$. For this construction, see Appendix 6.6.

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6 Appendix

6.1 Proof of Collapsing Sum

In Section **??**, we make use of the following computation:

Lemma *We have*

$$\sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) = \begin{cases} N^d & \textit{if } x = 0 \\ 0 & \textit{else} \end{cases}.$$

PROOF: In the case that $x = 0$, we have $\chi(x \cdot m) = \chi(0) = 1$. Since we are summing 1 over $\mathbb{Z}_N^d$, we get N^d. Now, suppose $x \neq 0$. We have $x_i \neq 0$ for some i. By the properties of the exponential,

$$\sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) = \prod_{j=1}^{d} \sum_{m_j \in \mathbb{Z}_N} \chi(m_j x_j).$$

Since $x_i \neq 0$, we have

$$\begin{aligned} \sum_{m_i \in \mathbb{Z}_N} \chi(m_i x_i) &= \sum_{m_i \in \mathbb{Z}_n} e^{-2\pi i m_i x_i / N} \\ &= \sum_{m_i = 0}^{N-1} (e^{-2\pi i x_i / N})^{m_i} \\ &= \frac{1 - (e^{-2\pi i x_i / N})^N}{1 - e^{-2\pi i x_i / N}} \\ &= 0. \end{aligned}$$

The third line follows from the geometric sum formula, given that $x_i \neq 0$. □

6.2 ∞-norm bound for Fourier Transform in $\mathbb{Z}_N^d$

In Section 2, we make use of the following result:

Lemma *If* $f : \mathbb{Z}_N^d \to \mathbb{C}$ *is a signal, then* $||\hat{f}||_\infty \leq N^{-d/2} ||f||_1$.

PROOF: By the definition of the Fourier transform,

$$\begin{aligned} |\hat{f}(m)| &= \left| N^{-d/2} \sum_{x \in \mathbb{Z}_N^d} \chi(-x \cdot m) f(x) \right| \\ &\leq N^{-d/2} \sum_{x \in \mathbb{Z}_N^d} |f(x)|. \end{aligned}$$

The last line is the desired quantity. □

6.3 Existence of argmin for Logan's Method

In proving Theorem 6 and Theorem 9, we use the following result:

Lemma *Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a signal and fix some set $S \subset \mathbb{Z}_N^d$. The set $\text{argmin}_{u \in A} ||u||_{L^1(\mathbb{Z}_N^d)}$ is nonempty. Here, A is the set of signals u such that $\hat{u}(m) = \hat{f}(m)$ for $m \notin S$.*

Proof: First, note that $\text{argmin}_{u \in A} ||u||_{L^1(\mathbb{Z}_N^d)}$ being nonempty is equivalent to the existence of $\min_{u \in A} ||u||_{L^1(\mathbb{Z}_N^d)}$. Further, if $||g||_{L^1(\mathbb{Z}_N^d)}$ is a minimum, we would have $|\hat{g}(m)| \leq N^{-d/2} ||f||_{L^1(\mathbb{Z}_N^d)}$. So, it suffices to show the existence of $\min_{u \in B} ||u||_{L^1(\mathbb{Z}_N^d)}$ where B is the subset of A consisting of u such that $|\hat{u}(m)| \leq N^{-d/2} ||f||_{L^1(\mathbb{Z}_N^d)}$ for all $m \in \mathbb{Z}_N^d$. Let D be the disk in $\mathbb{C}$ with radius $||f||_{L^1(\mathbb{Z}_N^d)}$ and let $\{X_1, \dots, X_{N^d}\}$ be an enumeration of $\mathbb{Z}_N^d$. Then, it suffices to show that the function $f : D^{N^d} \to \mathbb{R}_{\geq 0}$ given by

$$f(z_1, \dots, z_n) = \left|\left| \sum_{i=1}^{N^d} \chi(X_i \cdot x) z_i \right|\right|_{L^1(\mathbb{Z}_N^d)}$$

has a minimum. This follows from the fact that f is a continuous function. This completes the proof. □

6.4 Carleson's Theorem for the Unit Cube in $\mathbb{R}^d$

The purpose of this section is to prove a meaningful extension of Carleson's theorem on the real line to $\mathbb{R}^d$. As stated in [8], Carleson's theorem on the real line is the following:

Theorem (Carleson's Theorem in $\mathbb{R}$) *If $f \in L^2(\mathbb{R})$, then for almost all x we have*

$$f(x) = \lim_{N \to \infty} \int_{-N}^{N} e^{2\pi i \xi x} \hat{f}(\xi) d\xi.$$

One object studied in [8] is the Carleson operator, given by

$$\mathcal{C}f(x) = \sup_{N \in \mathbb{R}} \left| \int_{-\infty}^{N} e^{2\pi i \xi x} \hat{f}(\xi) d\xi \right|.$$

The proof of Carleson's theorem hinges on the estimate

$$\left| \left\{ x \in \mathbb{R}^d : \mathcal{C}f(x) > \lambda \right\} \right| \leq \frac{C ||f||_{L^2(\mathbb{R})}^2}{\lambda^2}.$$

Here, C is some uniform constant and $f \in L^2(\mathbb{R})$.

The technique that we will use to generalize into $\mathbb{R}^d$ was developed in [3] in the setting of Fourier series. The appropriate translation to the Fourier transform on $\mathbb{R}^d$ is the following:

Theorem (Carleson's Theorem in $\mathbb{R}^d$) *Let $P \subset \mathbb{R}^d$ be the convex hull of the points in $\mathbb{R}^d$ with each coordinate being*

± 1. *If $f \in L^2(\mathbb{R}^d)$, then for almost every x we have*

$$f(x) = \lim_{r\to\infty} \int_{rP} e^{2\pi i \xi\cdot x} \hat{f}(\xi) d\xi.$$

The altered version of the Carleson operator required for our proof is given by

$$\mathcal{D}f(x) = \sup_{r\in\mathbb{R}} \left| \int_{rP} e^{2\pi i \xi\cdot x} \hat{f}(\xi) d\xi \right|.$$

Here, $f \in L^2(\mathbb{R}^d)$. Correspondingly, our proof will hinge on the following:

Lemma *There is a uniform constant* C *such that for $f \in L^2(\mathbb{R}^d)$, we have*

$$\left| \left\{ x \in \mathbb{R}^d \, : \, \mathcal{D}f(x) > \lambda \right\} \right| < \frac{C||f||^2_{L^2(\mathbb{R}^d)}}{\lambda^2}.$$

PROOF: Our approach will be to split the integral

$$\int_{rP} e^{2\pi i \xi\cdot x} \hat{f}(\xi) d\xi$$

into the domains that are the convex hull a face of rP and the origin. Each of these domains will be handled identically. So, we consider

$$\int_0^r \int_{||\tau||_\infty < \xi_1} e^{2\pi i(\xi_1 x_1 + \tau\cdot(x_2,\ldots,x_d))} \hat{f}(\xi_1, \tau) d\tau d\xi. \tag{9}$$

Here, $\tau \in \mathbb{R}^{d-1}$. It will be easier to write this in terms of the function $g = (\hat{f} 1_S)^\vee$, where S is the union of the domains that we are integrating on above as $r \to \infty$. It will be important to note that $||g||_{L^2(\mathbb{R}^d)} \leq ||f||_{L^2(\mathbb{R}^d)}$. With this notation, we can write (9) as

$$\int_0^r \int_{||\tau||_\infty < \xi_1} e^{2\pi i(\xi_1 x_1 + \tau\cdot(x_2,\cdots,x_d))} \hat{g}(\xi_1, \tau) d\tau d\xi_1 = \int_0^r e^{2\pi i \xi_1 x_1} \tilde{g}(\xi_1, x_2, \ldots, x_d) d\xi_1.$$

This is just the Carleson operator on $g(\cdot, x_2, \ldots, x_d)$. So, for fixed $x_2, \ldots, x_d$ we have

$$\left| \left\{ x_1 \in \mathbb{R} \, : \, \sup_{r\in\mathbb{R}} |(9)| > \lambda \right\} \right| < \frac{C_1 ||g(\cdot, x_2, \ldots, x_d)||^2_{L^2(\mathbb{R})}}{\lambda^2}.$$

Integrating this over $x_2, \ldots, x_d$, we get

$$\begin{aligned} \left| \left\{ x \in \mathbb{R}^d \, : \, \sup_{r\in\mathbb{R}} |(9)| > \lambda \right\} \right| &< \frac{C_1 ||g||^2_{L^2(\mathbb{R}^d)}}{\lambda^2} \\ &\leq \frac{C_1 ||f||^2_{L^2(\mathbb{R}^d)}}{\lambda^2}. \end{aligned}$$

Doing this for all of the domains discussed above and taking the a large enough constant C, we have the desired result. □

We are now equipped to prove our extension of Carleson's theorem to $\mathbb{R}^d$. Our proof draws from Proposition 1.4 in [8].

PROOF: We would like to prove that

$$\limsup_{r\to\infty}\left|f(x)-\int_{rP}e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi\right|=0$$

for almost every x. Let g be a Schwartz function so that $||f-g||_{L^2(\mathbb{R}^d)}<\epsilon$. Then,

$$\begin{aligned}\left|f(x)-\int_{rP}e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi\right| &= \left|f(x)-g(x)+g(x)-\int_{rP}e^{2\pi i\xi\cdot x}(\hat{f}(\xi)-\hat{g}(\xi)+\hat{g}(\xi))d\xi\right| \\ &\leq |f(x)-g(x)|+\left|\int_{rP}e^{2\pi i\xi\cdot x}\widehat{f-g}(\xi)d\xi\right|+\left|g(x)-\int_{rP}e^{2\pi i\xi\cdot x}\hat{g}(\xi)d\xi\right|.\end{aligned}$$

So, we have

$$\limsup_{r\to\infty}\left|f(x)-\int_{rP}e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi\right|\leq|f(x)-g(x)|+\mathcal{D}(f-g)(x).$$

Now,

$$\left|x\in\mathbb{R}^d\,:\,|f(x)-g(x)|>\sqrt{\epsilon}\right|<\epsilon$$

and

$$\begin{aligned}\left|x\in\mathbb{R}^d\,:\,\mathcal{D}(f-g)>\sqrt{\epsilon}\right| &< \frac{C||f-g||^2_{L^2(\mathbb{R}^d)}}{\epsilon} \\ &= C\epsilon.\end{aligned}$$

So,

$$\left|\left\{x\in\mathbb{R}^d\,:\,\limsup_{r\to\infty}\left|f(x)-\int_{rP}e^{2\pi i\xi\cdot x}\hat{f}(\xi)d\xi\right|>2\sqrt{\epsilon}\right\}\right|<\epsilon+C\epsilon.$$

Taking $\epsilon\to0$, we see that $\int_{rP}e^{2\pi i\xi\cdot x}\hat{f}\xi d\xi$ converges to $f(x)$ as $r\to\infty$ for almost every x. □

6.5 Unique Recovery in S^1

In Section 4.2, we make use of the following result in the spirit of Theorem 4 of [2]:

Theorem *Let $f\,:\,S^1\to\mathbb{C}$ be a function such that $\hat{f}$ is supported on a finite set $W\subset\mathbb{Z}$. If f is known outside* *of $E \subset S^1$ and $|E||W| < \frac{1}{2}$, then f can be reconstructed exactly. Here, S^1 is given the surface measure such that $|S^1| = 1$.*

First, we have to establish an uncertainty principle. Suppose h is a nonzero function supported on a set E with finite Fourier support W. Then,

$$\begin{aligned} |h(x)| &= \left| \sum_{n \in W} e^{2\pi i x/n} \hat{h}(n) \right| \\ &\leq |W| ||h||_{L^1(S^1)}. \end{aligned}$$

Integrating both sides over E and dividing by the 1-norm of h, we get $1 \leq |E||W|$. We now proceed with the proof.

PROOF: Suppose g is function with Fourier support W' with $|W'| = |W|$ that agrees with f outside of E. For the sake of contradiction, assume that $g \neq f$. Let $h = f - g$. Then, the Fourier support of h has size at most $2|W|$ and the support of h is contained in E. By the uncertainty principle, we know that $2|W||E| \geq 1$, but this contradicts our assumption. $\square$

6.6 Construction of Nonlinear Fourier Series with Large Linear Fourier Support

Our goal is to construct sequences $\{F_n\}$ such that $\operatorname{supp}(\hat{b}) \sim \min\left(\operatorname{diam}(W) + 1, 2^{|W|-1}\right)$ where $\tilde{F} = (a, b)$. We will do this in both the case that $\min\left(\operatorname{diam}(W) + 1, 2^{|W|-1}\right) = \operatorname{diam}(W) + 1$ and the case that $\min\left(\operatorname{diam}(W) + 1, 2^{|W|-1}\right) = 2^{|W|-1}$.

We will handle the first case by induction on even and odd integers. Consider the sequence $\{F_n\}$ supported on $\{0\}$ where $F_0 = 1$. By (7), we have

$$\begin{aligned} \hat{b}(p) &= \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k+1} \\ \sum_{n=1}^{2k+1} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \leq n \leq 2k+1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \leq n \leq 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} \right) \\ &= \begin{cases} \frac{1}{\sqrt{2}} & \text{if } p = 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

If instead $\{F_n\}$ is supported on $[0, 1]$ with $F_0 = F_1 = 1$, then

$$\begin{aligned} \hat{b}(p) &= \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k+1} \\ \sum_{n=1}^{2k+1} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \leq n \leq 2k+1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \leq n \leq 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} \right) \\ &= \begin{cases} \frac{1}{\sqrt{2}} & \text{if } p = 0 \\ 0 & \text{else} \end{cases}. \end{aligned}$$

With the base cases established, assume the result holds for all $M < N$ where $N > 1$. For a sequence $\{F_n\}$ supported in $[0, N]$, we have

$$\hat{b}(N) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) F_N.$$

and

$$\hat{b}(0) = \left(\prod_{n=-\infty}^{\infty} (1 + |F_n|^2)^{-1/2} \right) F_0.$$

If $0 < p < N$, we write

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k+1} \\ \sum_{n=1}^{2k+1} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} \right) \\
&\quad = -F_N \overline{F_{N-1}} \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k-1} < N-1 \\ \sum_{n=1}^{2k-1} (-1)^{n+1} j_n = p-1}} \left(\prod_{\substack{1 \le n \le 2k-1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k-1 \\ n \text{ even}}} -\overline{F_{j_n}} \right) + F_N \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k} < N-1 \\ \sum_{n=1}^{2k} (-1)^{n+1} j_n = p-N}} \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ even}}} -\overline{F_{j_n}} \right) \\
&\qquad - \overline{F_{N-1}} \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k} < N-1 \\ \sum_{n=1}^{2k} (-1)^{n+1} j_n = p-N+1}} \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k \\ n \text{ even}}} -\overline{F_{j_n}} \right) + \sum_{k=0}^{\infty} \sum_{\substack{j_1 < \cdots < j_{2k+1} < N-1 \\ \sum_{n=1}^{2k+1} (-1)^{n+1} j_n = p}} \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ odd}}} F_{j_n} \right) \left(\prod_{\substack{1 \le n \le 2k+1 \\ n \text{ even}}} -\overline{F_{j_n}} \right).
\end{aligned}$$

By the inductive hypothesis,we can choose $F_1, \dots, F_{N-2}$ so that the sum in the first term is nonzero for all $0 < p < N$. After taking the minimum size of the first sum and the maximum size of the other sums over $0 < p < N$, and noting that the first term grows with respect to $|F_N F_{N-1}|$, while second, third, and fourth terms grow linearly or are constant with respect to $|F_N|$ and $|F_{N-1}|$, we can choose F_N and F_{N-1} large enough that $\hat{b}$ is nonzero on $[0, N]$.

We now turn our attention to a construction where $\min(\text{diam}(W) + 1, 2^{|W|-1}) = 2^{|W|-1}$. The key will be constructing a finite set $W \subset \mathbb{Z}$ such that

$$S(j_1, \dots, j_{2k+1}) = \sum_{n=1}^{2k+1} (-1)^{n+1} j_n.$$

is unique for each increasing set of numbers $j_1 < \cdots < j_n$ in W. It is simpler to construct this set by scaling so that $W \subset \mathbb{Q} \cap [0, 1]$. It is clear that we can recover a set of the original form by multiplying by the largest denominator in W.

Let W be the set of reciprocals of the first N primes. For distinct primes $p_1, \dots, p_{n+1}$, if $\frac{C}{p_1 \cdots p_n}$ is in simplified form for primes p_1, then

$$\frac{C}{p_1 \cdots p_n} \pm \frac{1}{p_{n+1}} = \frac{C p_{n+1} \pm p_1 \cdots p_n}{p_1 \cdots p_{n+1}}.$$

is in simplified form. This is because each prime divides exactly one of the terms in the numerator. Since there is only one increasing sequence in W containing a fixed set of prime reciprocals, there is only one sequence in W that gives an alternating sum with denominator in simplest form being the product of these primes. Thus, W has the desired property. In fact, it has a stronger property than needed because we only care about subsets of odd size.

We now scale W by the product of the first N primes. So, if $A = \{p_1, \ldots, p_N\}$ is the set of the first N primes, then

$$W = \left\{ \prod_{p \in B} p \, : \, B \subset A \text{ and } |B| = N - 1 \right\}.$$

We can scale this set further or choose N large enough so that $\operatorname{diam}(W) + 1 > 2^{|W|-1}$. Let $\{F_n\}$ be the sequence supported on W with value 1 on W. By (7) and the properties of W, we have that

$$|\operatorname{supp}(\hat{b})| = \sum_{\substack{k=0 \\ k \text{ odd}}}^{|W|} \binom{|W|}{k} = 2^{|W|-1}.$$