Diameter 2 Colorings

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1 Introduction

The question of whether diameter 2 graphs are 3-colorable in polynomial time has a well known open problem in graph theory. Diameter 2 graphs are graphs where each pair of vertices are either neighbors, or share a common neighbor. Subexponential colorings, polynomial time approximations, and polynomial time algorithms on certain classes (or graphs with restricted subgraphs) are all known. In this paper, we explore a number of methods for coloring diameter 2 graphs, and provide incremental results towards the goal of polynomial time 3 coloring for diameter 2 graphs.

2 Spectral Background

Matrices with respect to a graph will have a subscript denoting this. If context if obvious this subscript will be omitted.

Definition 1. *The adjacency matrix A has entry $a_{ij} = 1$ if there is an edge between vertices i, j and 0 otherwise.*

Definition 2. *The degree matrix D has entry $d_{ij} = deg(v_i)$ if $i = j$, and 0 otherwise.*

Definition 3. *The vertex-edge incidence matrix R of a graph G is $n \times m$, with entry $r_{ij} = 1$ if vertex v_i is an endpoint of edge e_j, and 0 otherwise.*

Definition 4. *Laplacian L is defined as A-D.*

Definition 5. *The signless Laplacian $|L|$ is defined as A+D. Note this matrix is positive semidefinite.*

Lemma 2.1. *If R is the vertex-edge incidence matrix of a graph G, then $RR^T = A + D$, and $R^T R = A_{L(G)} + 2I_{|E|}$*

Proof. The first follows from observing that rows multiplied with a transpose of itself will count all edges adjacent to a vertex (which is its degree), and rows multiplied with transposes of other rows will yield a 1 if they share an edge, and

0 if not. Since we are dealing with simple graphs, we will not count more than one edge.

The second follows similarly, by noting that each edge connects exactly 2 vertices, so the "degree matrix" is $2I_{|E|}$. □

Definition 6. *The characteristic polynomial of the adjacency matrix of a graph G is denoted by $P_G(\lambda)$*

Definition 7. *The characteristic polynomial of the signless Laplacian of a graph G is denoted by $Q_G(\lambda)$*

Definition 8. *Two graphs are called* cospectral *if they have the same spectrum.*

Definition 9. *Two graphs are called* Q-spectral *if they have the same polynomial $Q(\lambda)$*

Definition 10. *For a graph G, we define the line (dual) graph G' to swap vertices and edges.*

Definition 11. *Two graphs are called* L-spectral *if their line graphs are cospectral*

Lemma 2.2. $P_{G+cI}(\lambda) = P_G(\lambda + c)$

Proof. Note that $(\det((A + cI) - \lambda I) = \det(A + (c - \lambda)I)$, then complete a change of variable.

□

Lemma 2.3. *Let $B \in \mathbb{R}^{n\times m}$, with $n \leq m$ and $\operatorname{rank}(B) = r$. Then,*

$$\det(\lambda I_m - B^\top B) = \lambda^{m-n} \det(\lambda I_n - BB^\top)$$

Proof. Let $M = B^\top B \in \mathbb{R}^{m\times m}$, and $N = BB^\top \in \mathbb{R}^{n\times n}$.Then, both of these are positive semidefinite, and share the same nonzero eigenvalues. As a result, $\operatorname{rank}(M) = \operatorname{rank}(N) = r$. To see this, observe that for any vector $x \in \mathbb{R}^m$,

$$x^\top Mx = x^\top B^\top Bx = (Bx)^\top (Bx) = \|Bx\|^2 \geq 0,$$

so M is positive semidefinite. Similarly, N is positive semidefinite.

Now, suppose $\lambda \neq 0$ is an eigenvalue of M with eigenvector $x \in \mathbb{R}^m$. Then:

$$B^\top Bx = \lambda x.$$

Apply B to both sides:

$$BB^\top (Bx) = \lambda(Bx).$$

So either $Bx = 0$, which would imply $\lambda = 0$, a contradiction, or $Bx \neq 0$, and Bx is an eigenvector of N with eigenvalue λ. A similar argument shows that every nonzero eigenvalue of N is also an eigenvalue of M.

Then, let the nonzero eigenvalues of $B^\top B$ and $BB^\top$ be $\mu_1, \ldots, \mu_r$. We have:

$$\det(\lambda I_m - B^\top B) = \prod_{i=1}^{r}(\lambda - \mu_i) \cdot \lambda^{m-r}$$

$$\det(\lambda I_n - BB^\top) = \prod_{i=1}^{r}(\lambda - \mu_i) \cdot \lambda^{n-r}$$

Therefore,

$$\det(\lambda I_m - B^\top B) = \lambda^{m-n} \det(\lambda I_n - BB^\top)$$

□

Lemma 2.4. *[Cvetkovic]* $P_{L(G)}(\lambda) = (\lambda+2)^{m-n} Q_G(\lambda+2)$, *and if Spec(G)* $=$ $\lambda_1, \ldots, \lambda_t$, *then the eigenvalues of L(G) are* $\theta_i = \lambda_i - 2$ $i \in [t]$, $\theta_i = -2$ *else.*

Proof. This follows from 2.1 and 2.3. □

Theorem 2.5 (Cvetkovic)**.** *Q-spectral implies L-spectral*

Proof. If graphs are $Q-$spectral, then they have the same number of vertices and edges. Then, L-spectrality follows from 2.4 □

However, note that we can find graphs that share a line graph Laplacian that are not $Q-$spectral. To see this, consider $K_4 - e \cup K_2$ and the butterfly graph with an additional disconnected vertex. Then, both line graphs have characteristic polynomial $\lambda(\lambda^2 - \lambda - 4)(\lambda - 1)(\lambda + 1)^2$, however the first has Q polynomial $\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^2 - 5\lambda + 2)$, while the latter has $\lambda^2(\lambda - 1)(\lambda-2)(\lambda-3)(\lambda^2 - 5\lambda + 2)$.

Lemma 2.6. *An independent set in* $L(G)$ *corresponds to a matching in* G*.*

Proof. By selecting edges in a matching, we ensure that these edges are adjacent to no other edges. Thus, these edges are not connected in the line graph. □

Lemma 2.7. *For a graph* G, $|E| = -\frac{p_1}{2}$ *where* p_1 *is the coefficient of* λ^{n-1} *in* Q_G

Proof. The trace of $|L|$ is equal to the sum of vertex degrees of G, and all lengths of walk 2 are edges. □

Theorem 2.8 (Cvetkovic et al.)**.** *Let* H *be* G *after switching edges* ab, cd *to non edges* ad, bc*. Let* x *be a principal eigenvector of* G*. If* $(x_a - x_c)(x_b - x_d)] \geq 0$*. then* $\lambda_1^H \geq \lambda_1^G$*, with equality* iff *both of the two products are 0.*

Proof. First, note that

$$\lambda_1^G = \sup_{x \in R^n - \{0\}} \frac{x^T Q^G x}{x^t x} = \max_{\|x\|=1} x^T Q x$$

, then if we change edges we have, if x is the eigenvector corresponding to λ_1^G, that

$$\lambda_1^H - \lambda_1^G = \max_{\|y\|=1} y^T Q^H y - x^T Q^G x \geq x^T (A^H - A^G) x + x^T (D^H - D^G) x$$

With equality if and only if x is also the principle eigenvector for Q^H. Then, from this we obtain

$$\lambda_1^H - \lambda_1^G \geq 2(x_a - x_c)(x_b - x_d)$$

and our result follows. $\square$

Theorem 2.9. *Let λ_1 be the largest eigenvalue of A. Then,*

$$\lambda_1 \geq \frac{2m}{n} \tag{1}$$

Proof. Consider that $\lambda_1 = \max_{x \in R^n} \frac{x^T A x}{x^T x}$, and let $x = 1_n$. Then, we count each edge twice in the numerator, yielding $2m$ and the denominator will be n summations of 1, so we have our desired lower bound. $\square$

Theorem 2.10 (Ando)**.** *If G is vertex diameter 2-critical with $\delta \geq 4$ then*

$$\frac{5n - 17}{2} \leq m \leq \frac{5n + 17}{2} \tag{2}$$

Proof. If $\delta \geq 5$, then we have $deg(V) \geq 5n$. Else, if there is a vertex of degree 4, we have

$$\begin{aligned} deg(V) &= deg(t) + deg(N(t) + deg(V - N[t])) \qquad (3) \\ &\geq 4 + (n-1) + 4(n-5) \qquad (4) \\ &= 5n - 17 \qquad (5) \end{aligned}$$

$\square$

Theorem 2.11. *Let D be the distance matrix, $\bar{A}$ the complement and J of all ones. Then,*

$$\begin{aligned} D^2 &= 4(J - I)\bar{A} + A^2 \qquad (6) \\ D &= 2(J - I) + A \qquad (7) \end{aligned}$$

Lemma 2.12. *Given a star graph $K_{1,s}$, the line graph $L(K_{1,s})$ has a hamiltonian cycle.*

Proof. Since all edges are adjacent to the central vertex (and thus each other), we have that $L(K_{1,s}) = K_s$. $\square$

Theorem 2.13. *For all integers $n \geq 4$, the line graph of the complete graph K_n has a Hamiltonian cycle.*

Proof. Assign the vertices some arbitrary order. Then, traverse all edges of v_1 in any order such that we traverse the edge connected v_1 to v_2 last. Then, traverse the edges of v_2 similarly, traversing the edge connecting v_2 to v_3 last. $\square$

Definition 12. *A* circuit *C is a walk in which all edges are distinct, and the first and last vertices are equal. A $D-$circuit is a circuit in which every edge of G is incident with at least one vertex in the $D-$circuit.*

Definition 13. *If C is a circuit and Z a cycle of G such that $V(Z) \cap V(C) \neq \emptyset \neq V(Z) \cap (V(G) - V(C))$ and $G[E(C) \triangle E(Z)]$ (where $\triangle$ denotes symmetric difference) is connected, then Z is called a C-*augmenting cycle*. Clearly, if C is a circuit and Z is a C-augmenting cycle, then $G[E(C) \triangle E(Z)]$ is also a circuit, and $|V(G[E(C) \triangle E(Z)])| > |V(C)|$, implying that if C is a maximal circuit, we have no $C-$augmenting cycle.*

Definition 14. *Define τ to be P_3 with two new non-adjacent vertices adjacent to one of the end vertices. Similarly, define τ^+ to be P_3 with two new adjacent vertices adjacent to one of the end vertices.*

Lemma 2.14 (Veldman 1988)**.** *Let G be a graph and C a maximal circuit. Then, there is no cycle Z with*

$$V(Z) \cap V(Z) \neq \emptyset \neq V(Z) \cap (V(G) - V(C)) \wedge |E(z) \cap E(C)| \leq 1 \tag{8}$$

Definition 15. *A* block *of a graph is a maximal connected subgraph with no* cut-vertices*, that is no vertices whose removal would disconnect the graph. For example, all complete graphs of size 3 or larger are blocks, while no paths of length 3 or longer are.*

Lemma 2.15 (Veldman 1988)**.** *If $x_1, \ldots, x_k$ is a path in G, and for $1 \leq i < j < k$ if we have $x_i x_{i+1}, x_j x_{j+1}$ in the same block B of G, then $x_m x_{m+1} \in E(B) \forall i < m < j$.*

Theorem 2.16 (Veldman 1988)**.** *Let G be a connected graph that is not a tree such that every subgreaph isomorphic to τ or τ^+ with $d(a_i) \geq 2, i \in \{1,2\}$ satisfies at least one of the following:*

1. *$|N(a) \cap N(c)| \geq 2$*
2. *$|N(b) \cap N(c) \geq 1|$*
3. *For $i \in \{1,2\}, |N(b) \cap N(a_i)| \geq \begin{cases} 2 & a_1 a_2 \in E(G) \\ 3 & else \end{cases}$*

4. For $i \in \{1, 2\}, |N(a) \cap N(b)| \geq 1$

Then $L(G)$ is hamiltonian

Proof. Let G satisy our initial conditions, but assume $L(G)$ is nonhamiltonian. Let C be a maximal circuit of G. Then C is not a D-circuit of G, as by Harary and Nash-Williams G would otherwise have hamiltonian L(G). Thus there exists a path u_1uu_2u with $u_1, u_2 \notin V(C)$ and $u \in V(C)$. Let $vv_1, vv_2 \in E(C)$. Then, as a result of 2.14 it follows that some subgraph $H \leq G$ with vertices $\{u_1, u_2, u, u_1, u_2\}$ is isomorphic to τ or τ^+, and $(N(u_1) \cap N(u)) - \{u_2\} = N(u_2) \cap N(u) = \emptyset$. . We must have $d_C(u_1) \geq d_C(u_2) \geq 2$ $(i = 1, 2)$, so we know that H satisfies (3) or (4).

First, assume that H satisfies (3). Then, assume $H = \tau^+$, so $v_1v_2 \in E(G)$. Assume without loss of generality that u_2 and v_1 have a common neighbor w with $w \neq u$. By 2.14, $v_1w \in E(C)$. If $v_1v_2 \in E(C)$, then vu_2wv_1v is a C-augmenting cycle, a contradiction with 13. If $v_1v_2 \in E(G) - E(C)$, then $vu_2wv_1v_2v$ is a C-augmenting cycle, which contradicts that C is maximal.

Thus, we can only have that $u_1u_2 \notin E(G)$, so $H = \tau$. Assume without loss of generality that v_1 and u_2 have common neighbors not equal to v, label these w_1 and w_2. By 2.14, $v_1w_1, v_1w_2 \in E(C)$. If v_1v, v_1w_1, and v_1w_2 are in the same block of C, then $C - \{v_1w_1, v_1w_2\}$ is connected, implying that $u_2w_1v_1w_2u_2$ is a C-augmenting cycle, contradicting 13. If v_1v and v_1w_1 are in different blocks of C, then $C - \{v_1v, v_1w_1\}$ is connected (since every block of C is 2-edge-connected), so $vu_2w_1v_1v$ is a C-augmenting cycle, which contradicts that C is maximal.

So H cannot satisfy (3), and instead must satisfy (4).

Call a path P *special* if it satisfies the following requirements:

- P has origin v,
- $E(P) \subseteq E(C)$,
- each block of C contains at most one edge of P, and
- u_1 and the terminus of P have a common neighbor.

Note that, if P is a special path, then, by the third requirement, $C - E(P)$ is connected.

Since H_1 satisfies (4), G contains a special path of length 1. Let P be a special path of maximum length, x the terminus of P, y the immediate predecessor of x on P, and z a common neighbor of u_1 and x.Then , we know $z \notin V(P)$, otherwise G contains the C-augmenting cycle $Q_1 \cup vu_2u_1z$, where Q_1 denotes the (v, z)-subpath of P. Also, $z \neq u_2$, otherwise $P \cup vu_2x$ is a C-augmenting cycle. Furthermore, xz is an edge of C, otherwise the cycle Z_1, with $Z_1 = P \cup uu_2u_1zx$ is a C-augmenting cycle. Moreover, the edges xy and xz are in the same block of C; assuming the contrary, by 2.15, all edges of $E(P) \cup \{xz\}$ are in different blocks of C, again yielding the contradiction that Z_1 is a C-augmenting cycle. This contradiction is avoided only if $\{xy, xz\}$ is a 2-edge cut of C. Thus, either

$d_C(x) = 2$ or x is a cut vertex of C. If $d_C(x) = 2$, then $G[E(C)\triangle E(Z_1)]$ consists of a trivial component and a component that is a circuit; the latter circuit contains one vertex more than C, contradicting the maximality of C. Thus x is a cut vertex of C.

Let B be a block of C containing x and different from the block that contains xy (and xz). Then, by 2.15, B differs from all blocks of C that contain an edge of P. Let xx_1 and xx_2 be two edges of B and let $H_1 = G[\{u_1, z, x, x_1, x_2\}]$. By 2.14, $u_1x \notin E(G)$. Also, $u_1x_i \notin E(G)$, otherwise $P \cup vu_2u_1x_ix$ is a C-augmenting cycle ($i = 1, 2$). Since P is a longest special path, $zx_i \notin E(G)$ ($i = 1, 2$), so H_1 is isomorphic to τ or τ^+. Since $d_G(x_i), d_C(x_i) \geq 2$ ($i = 1, 2$), H_1 satisfies one of our requirements. We finish by showing that each case yields a contradiction.

First suppose H_1 satisfies (1). Let $z_1 \in (N(u_1) \cap N(x)) - \{z\}$. As in the last paragraph, we have $z_1 \notin V(P) \cup \{u_2\}$, xz_1 is an edge of C, and xz_1 and xy are in the same block of C. Since xz is also in this block, $C - (E(P) \cup \{xz\})$ is connected, and Z is a C augmenting cycle, which is a contradiction.

Now suppose H_1 satisfies (2). Let $y_1 \in N(z) \cap N(x)$. As a result of 2.14, $y_1 \notin \{u_1, u_2\}$. If $y_1 \in V(P)$, then $Q_2 \cup vu_2u_1zy_1$, with Q_2 as the (u, y_1)-subpath of P, is a C-augmenting cycle, whether $zy_1 \in E(C)$ or not. Thus we must have $y_1 \notin V(P)$. If xy_1 and zy_1 are edges of C, then Z_1 is a C-augmenting cycle, else $P \cup vu_2u_1zy_1x$ forms a contradiction.

Next suppose H_1 satisfies (3), with $b = z$ and $x_1, x_2 = a_1, a_2$. Let $x_3 \in (N(z) \cap N(x_1)) - \{x\}$. Earlier arguments imply that x_3 cannot be a vertex in $V(P) \cup \{u_1, u_2\}$. We consider all possible cases for the membership of x_1x_3, x_3z with respect to C. If both x_1x_3 and x_3z are edges of C, then Z_1 is a C-augmenting cycle. If both x_1x_3 and x_3z are in $E(G) - E(C)$, then the cycle Z_1, with $Z_1 = P \cup vu_2u_1zx_3x_1x$ is a C-augmenting cycle. Assume $x_1x_3 \in E(G) - E(C)$ and $x_3z \in E(C)$. By 2.15, x_3z is not an edge of the block B of C containing x_1, so, x_3z is not a cut edge of the connected subgraph $(C + x_1x_3) - (E(P) \cup \{xx_1\})$ of G, since $zx \cup Q_3 \cup x_1x_3$ (Q_3 is an (x, x_1)-path in $B - xx_1$) is a (z, x_1)-path in this subgraph. This implies Z_1 is a C-augmenting cycle. Finally, Now assume $x_1x_3 \in E(C)$ and $x_3z \in E(G) - E(C)$. We finally distinguish two cases.

First, if $x_1x_2 \in E(G)$, then $x_3 \neq x_2$, otherwise H_1 would satisfy (2), which we already proved generates a contradiction. If $x_1x_2 \in E(C)$, then Z_1 is a C-augmenting cycle. If $x_1x_2 \in E(G) - E(C)$, then $P \cup uu_2u_1zx_3x_1x_2x$ is a C-augmenting cycle, so we must have no edge x_1x_2.

Thus, we must have that $x_1x_2 \notin E(G)$. Consider some $x_4 \in (N(z) \cap N(x_1)) - \{x, x_3\}$. Similar to the consideration for x_3, we may assume $x_4 \notin V(P) \cup \{u_1, u_2\}$, $x_1x_4 \in E(C)$, and $x_4z \in E(G) - E(C)$. If both x_1x_3 and x_1x_4 are edges of B, then $B - \{xx_1, x_1x_3\}$ is connected and Z_1 is a C-augmenting cycle. Else, with $x_1x_4 \notin E(B)$, then by 2.15 all edges of $E(P) \cup \{xx_1, x_1x_4\}$ are in different blocks of C, and hence $P \cup vu_2u_1zx_4x_1x$ is a C-augmenting cycle.

Finally, suppose H_1 satisfies (iv). Assume without loss of generality that $N(u_1) \cap N(x_1) \neq \emptyset$. Then $P \cup xx_1$ is a special path longer than P, our final contradiction.

$\square$

Theorem 2.17 (Veldman 1988)**.** *If G is diameter 2 with at least 4 vertices, L(G) is Hamiltonian*

Proof. If G has diameter 1, then G is complete, so L(G) is hamiltonian by 2.13. Else, every induced subgraph isomorphic to τ or τ^+ satisfies 2.16, so either L(G) is hamiltonian or G is a tree. If G is a tree, then G is isomorphic to the star graph, and L(G) is hamiltonian by 2.12 $\square$

3 Removals and Interlacing

Theorem 3.1 (Cauchy, Poincare)**.** *Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric. Let $B \in \mathbb{R}^{m\times m}$ with $m < n$ be a principal submatrix (obtained by deleting both the i-th row and i-th column for some i). Suppose A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and B has eigenvalues $\mu_1 \leq \cdots \leq \mu_m$. Then:*

$$\lambda_k \leq \mu_k \leq \lambda_{k+n-m} \quad \textit{for } k = 1, \ldots, m.$$

Proof by Williamson. Without loss of generality, assume

$$A = \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix}.$$

Let $\{x_1, \ldots, x_n\}$ be the eigenvectors of A, and $\{y_1, \ldots, y_m\}$ the eigenvectors of B.

Define the following subspaces:

$$V = \operatorname{span}(x_k, \ldots, x_n), \quad W = \operatorname{span}(y_1, \ldots, y_k),$$

$$W' = \left\{ \begin{bmatrix} w \\ 0 \end{bmatrix} \in \mathbb{R}^n \,\middle|\, w \in W \right\}.$$

Since $\dim(V) = n-k+1$ and $\dim(W') = \dim(W) = k$, there exists $w' \in V \cap W'$, and $w' = \begin{bmatrix} w \\ 0 \end{bmatrix}$ for some $w \in W$. Thus,

$$w'^T A w' = \begin{bmatrix} w^T & 0 \end{bmatrix} \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix} \begin{bmatrix} w \\ 0 \end{bmatrix} = w^T B w.$$

Since we have $\lambda_k = \min_{x\in V} \frac{x^TAx}{x^Tx}, \quad \beta_k = \max_{x\in W} \frac{x^TBx}{x^Tx}$, we can conclude

$$\lambda_k \leq \frac{w'^T A w'}{w'^T w'} = \frac{w^T B w}{w^T w} \leq \beta_k.$$

To prove the second inequality, define:

$$V = \operatorname{span}(x_1, \ldots, x_{k+n-m}), \quad W = \operatorname{span}(y_k, \ldots, y_m),$$

$$W' = \left\{ \begin{bmatrix} w \\ 0 \end{bmatrix} \in \mathbb{R}^n \;\middle|\; w \in W \right\}.$$

Then $\dim(V) = k+n-m$ and $\dim(W') = m-k+1$, so there exists $w' \in V \cap W'$ and $w' = \begin{bmatrix} w \\ 0 \end{bmatrix}$ for some $w \in W$. We again compute $w'^T A w' = w^T B w$, therefore,

$$\lambda_{k+n-m} = \max_{x \in V} \frac{x^T A x}{x^T x} \geq \frac{w'^T A w'}{w'^T w'} = \frac{w^T B w}{w^T w} \geq \min_{x \in W} \frac{x^T B x}{x^T x} = \beta_k.$$

□

4 Eigenvalues to Partition Certain Graphs

Lemma 4.1. *Reordering vertices preserves the spectrum of the adjacency, laplacian, signless laplacian, and line graph.*

Proof. To see this, reorder the eigenvectors in the same way the matrix was reordered. □

Theorem 4.2 (Aspvall and Gilbert 1983)**.** *Let G be a block regular* 3- *partite graph. Then there is a set of 2 eigenvectors whose sign patterns properly partition vertices of the graphs.*

Proof. Let G be tripartite with partition V_1, V_2, V_3 with each part having size r, s, t respectively. Then, let

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{12}^T & 0 & A_{23} \\ A_{13}^T & A_{23}^T & 0 \end{bmatrix} B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}$$

Where A is the adjacency matrix of G, and B is the block degree matrix. Observe that these share eigenvalues, up to multiplications by 1_i to scale the block degree eigenvector to match the dimension of A (and vice versa), that is $(\alpha 1, \beta 1, \gamma 1)^T$ is an eigenvector of A with eigenvalue λ if and only if (α, β, γ) is an eigenvector of B with eigenvalue λ.

We apply the following coloring algorithm: Repeatedly select and eigenvalue and use the sign of its components (let zero be positive) to refine the coloring so that all vertices with the same signs across multiple eigenvectors will be in the same color class.

We can see that if we use every eigenvalue, we will partition each vector into its own set. Aspvall and Gilbert conjecture that this algorithm will find a coloring after only considering negative eigenvalues. Furthermore, in our block regular graph, we need only 2.

Note that $rb_{12} = sb_{21}, rb_{13} = tb_{31}, and sb_{23} = tb_{32}$, then let D be a diagonal matrix with entries $\sqrt{r}, \sqrt{s}, \sqrt{t}$. Then, let $B' = DBD^{-1}$, which is symmetric and nonnegative with zero diagonal entries, and furthermore B and B' share eigenvalues, with identical sign patterns. These then partition the rows of B' into singletons, and thus B and A.

This algorithm does not provide the needed eigenvectors, but they can be found in polynomial time, unless one of B's negative eigenvalues has higher multiplicity as an eigenvalue of A than of B. Otherwise, we need not try all pairs of eigenvalues. The negative sum od the corresponding eigenvalues is equal to the spectral radius of A, so we need only try half as many colorings as there are negative eigenvalues of its adjacency matrix.

Note that the conjecture made has, to the best of my knowledge, not been proved. This conjecture is also a critical part of the proof of the Alon-Kahale algorithm. □

4.1 Alon-Kahale Algorithm

The Alon-Kahale assumes the graph is random (edges are added between 3 groups of n vertices with uniform probability p), that we remove high degree vertices (with degree greater than or equal to $5d$, where $d = np$), and that color classes are balanced (shown by each color class having n vertices). This is because we wish to control the spectrum, in order for proofs about eigenvalue partitions to hold. However, diameter 2 graphs tend to have a relatively controlled spectrum.

The algorithm roughly works by (after removing high degree vertices) first finding a linear combination of the smallest two eigenvalues of the adjacency matrix with median zerom normalized to have ℓ_2 norm $\sqrt{2n}$, call this t_u. Then, it partitions vectors to be all vertices by norm after multiplication by t_u into three categories - within $\left|\frac{1}{2}\right|$ of 0, strictly greater than $\frac{1}{2}$, and strictly less than $-\frac{1}{2}$. This first stage almost always generates a proper 3 coloring. It then performs a balancing stage, by recoloring algorithms by the least popular color of its neighbors. After this stage, it takes all vertices with few neighbors colored any one color, and attempts to brute force color these final vertices. With high probability, this produces a proper 3 coloring.

Empirical experimentation shows that the relative size of the color classes are not so important, nor is the exact probability p, although they must both be so that no one color class is "too small", so any failed coloring can usually be resolved by keeping the same relative sizes of color classes and probability p, and increasing n. This is because large imbalance leads to certain thresholds in the later stages not being met, and the algorithm terminating (even though the first stage usually produces a very close to correct 3-coloring).

I found that removing high degree vertices was not quite so easily relaxed, and that leaving high degree vertices tended to lead to a failure in coloring. This is because, intuitively, these vertices have more impact on the spectrum, and so they are "overrepresented" in the spectrum.

I also found that given an arbitrary random graph with unknown probability p and unknown color class sizes, by observing the degree distribution we are able to estimate the expected number of vertices in each color class. This informs balancing in later stages of the algorithm.

I made efforts to control the spectrum of the graph, and to attempt to reduce the degree of high degree vertices. Observe that if we have a triangle lattice, we can add or remove edges from this lattice without affecting coloring, as some vertices are forced to be different (even if they are far). Alternatively, if two vertices are at odd distance from each other along some cycle, then we can remove an edge between them and know that they will still be different colors.

5 Degree-Based algorithms

5.1 Brute Force Dominating Set

Let G be a graph with minimum degree $\geq \alpha n$. Let $\log(n) = \alpha n$. Then, select $\log(n)$ vertices, call this set S. We can color S using brute force colorings in time $3^{\log n} = O(n)$. Consider an arbitrary vertex, and consider the odds that this vertex or none of its neighbors have been colored, with the assumption this node has $\log n$ neighbors. We are have $\binom{n}{\log n}$ total possible options for S , and $\binom{n-\log n}{\log n}$ of these will avoid the $\log n$ neighbors of our vertex. Then, we bound the odds that vertex i will not be in $N[S]$ as follows:

$$P(i \notin N[S]) \leq \frac{\binom{n-\log n}{\log n}}{\binom{n}{\log n}} \tag{9}$$

$$= \frac{(n-\log n)!^2}{(n-2\log n)!n!} \tag{10}$$

By Stirling

$$\approx \frac{\left(\sqrt{2\pi(n-\log n)}\left(\frac{n-\log n}{e}\right)^{n-\log n}\right)^2}{\sqrt{2\pi(n-2\log n)}\left(\frac{n-2\log n}{e}\right)^{n-2\log n}\cdot\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \tag{11}$$

$$\approx \frac{\left(\frac{n-\log n}{e}\right)^{2(n-\log n)}}{\left(\frac{n-2\log n}{e}\right)^{n-2\log n}\left(\frac{n}{e}\right)^n} \tag{12}$$

$$\approx_{n\to\infty} \left(\frac{n-\log n}{n}\right)^n \tag{13}$$

So this approach does not quite work.

A second approximation considers the odds a vertex with $\log n$ neighbors has any of these $\log n$ neighbors selected. The odds the first neighbor is not selected is $\left(1-\frac{\log n}{n}\right)$, and since each neighbor is independent, we have our end result as $\left(1-\frac{\log n}{n}\right)^{\log n}$. This goes to 1 as $n \to \infty$. So we cannot prove our result with this method. However, we have parameters. If A = the number of neighbors that our vertex has, we consider our probability as $\lim_{n\to\infty}\left(1-\frac{\log n}{n}\right)^A$.

Select one vertex of maximum degree $c\log n$, color this vertex and its neighborhood by brute force (time $3^{c\log n} = O(n3^c)$, then color the remaining neighbors by 2-list coloring. However, we are not guaranteed a vertex with small degree.

If we have $\log n$ vertices with degree $\frac{cn}{\log n}$, then the odds that any vertex is not colored approaches is $\left(1-\frac{c}{\log n}\right)^{\log n} \to \frac{1}{e^c}$ as n grows large. Then, the odds that at least one of our $n-\log n$ vertices are not in a neighborhood is $1-(1-e^{-c})^{n-\log n}$, which approaches 1 as $n\to\infty$, so this does not work either.

Say we, similarly, take $f(n)$ samples of $\log n$ nodes with degree $\frac{n}{\log n}$. Then, the odds that all nodes are hit in a sample is $\left(1-\left(\frac{n-\frac{n}{\log n}}{n}\right)^{\log n}\right)^{n-\log n}$. Then, the odds that 1 sample fails is 1 minus this. So the probability that $f(n)$ samples fail is $\left(1-\left(1-\left(\frac{n-\frac{n}{\log n}}{n}\right)^{\log n}\right)^{n-\log n}\right)^{f(n)}$, which goes to 1 for polynomial f, and thus we can say with probability 1 all samples will fail, and we will not obtain a dominating set.

Thus, we cannot naively select high degree nodes. The diameter 2 constraint should give us some power here, we have P(neighbor or share a neighbor) = 1, so if we can calculate 2/3 of P(neighbor), P(share a neighbor), and P(neighbor and shares a neighbor) then we can get the final third for free.

5.2 Size of Dominating Set

Theorem 5.1 (Henning and Yeo). *Dominating number of diameter 2 graphs is bounded above by* $\left(\frac{1+\ln(\delta)}{\delta}\right)n$

Theorem 5.2 (Henning nad Yeo). *For diameter 2 graphs, if the min degree is larger than* $\ln(n)\sqrt{n}$*, then the size of the dominating set is bounded by* $1+\sqrt{n}$

Proof. Assume $n\geq 3$. Then, let our domination number be $\gamma_t(G)$. We have by 5.1 abd by noting that $\frac{1+\ln(n)}{n}$ is decreasing in n, that

$$\gamma_t(G) \leq \left(\frac{1+\ln(\delta)}{\delta}\right)n \tag{14}$$

$$\leq \left(\frac{1+\ln(\sqrt{n}\ln(n))}{\sqrt{n}\ln(n)}\right)n \tag{15}$$

$$= \left(\frac{1+\frac{1}{2}\ln(n)+\ln(\ln(n))}{\ln(n)}\right)\sqrt{n} \tag{16}$$

$$= \frac{\sqrt{n}}{2}+\frac{1+\ln(\ln(n))}{\ln(n)}\sqrt{n} \tag{17}$$

So we show that $\frac{1+\ln(\ln(n))}{\ln(n)}\sqrt{n} \leq 1 + \frac{\sqrt{n}}{2}$. This holds for $n \geq 213$ as the LHS is decreasing and less than $\frac{1}{2}$, and computer verification resolves all other cases. $\square$

Theorem 5.3 (Henning and Yeo)**.** *For diameter 2 graphs, the dominating set is bounded by $1 + \sqrt{n\ln(n)}$, and this bound is tight.*

Proof. We set up a similar inqeuality to 5.2

$$\gamma_t(G) \leq \left(\frac{1+\ln(\delta)}{\delta}\right) n \tag{18}$$

$$\leq \left(\frac{1+\ln(\sqrt{n\ln(n)})}{\sqrt{n\ln(n)}}\right) n \tag{19}$$

$$= \left(\frac{1+\frac{1}{2}\ln(n)+\ln(\ln(n))}{\sqrt{\ln(n)}}\right)\sqrt{n} \tag{20}$$

$$= \frac{\sqrt{n\ln(n)}}{2} + \frac{1+\ln(\ln(n))}{\ln(n)}\sqrt{n\ln(n)} \tag{21}$$

So we need prove that

$$\left(\frac{1+\frac{1}{2}\ln(\ln(n))}{\ln(n)}\sqrt{n\ln(n)} \leq 1 + \frac{\sqrt{n\ln(n)}}{2}\right) \tag{22}$$

Which holds when $n \geq 24$, as the LHS is decreasing and less than $\frac{1}{2}$. The following cases follow from computer verification. $\square$

Theorem 5.4 (Henning and Yeo)**.** *If the domination number is greater than $1 + \sqrt{n}$, then the min. degree is between $\sqrt{n}$ and $\ln(n)\sqrt{n}$*

Proof. This follow from 5.2 and by noting that if we have minimum degree of some vertex v less than $\sqrt{n}$, we can take $N(v)$ as a dominating set with size less than or equal to $1 + \sqrt{n}$ $\square$

Thus, even if min degree is $\frac{n}{\ln n}$, we have the size of the dominating set bounded by $\ln^2(n)$, which is too large to brute force color.

5.3 Claw Free Dominating Set

Definition 16. *The* claw *graph is $K_{1,3}$.*

Definition 17. *A vertex v is* complete *so a set S if it is adjacent to every vertex in S, and* anticomplete *if it is adjacent to none of the vertices in S.*

Definition 18. *A* W-join *is a pair of disjoint non-empty sets of vertices (A,B) such that $|A|+|B| > 2$, both are cliques, A is neither complete nor anticomplete to B, and every vertex of $V(G)\backslash(A \cup B)$ is either complete or anticomplete to A and complete or anticomplete to B.*

Definition 19. *Neighborhoods of vertices u, v are* distinct *if we do not have $N[u] \subset N[v]$, or vice versa. If all such vertices have this property in a graph, we say the graph has* distinct neighborhoods.

Definition 20. *A* proper circular arc graph *is defined by the following characteristics: each vertex corresponds to an arc on a circle, and vertices are adjacent if their arcs intersect. Furthermore, what makes this graph proper is that no arc is entirely contained within another.*

Theorem 5.5 (Martin et al.)**.** *Every claw free graph with distinct neighborhoods, maximal independent set number at least 3, and more than 13 vertices is either a proper circular-arc graph or a line graph.*

Theorem 5.6 (Bouqet et al.)**.** *Finding a dominating set is polynomial time solvable for line graphs with diameter 2.*

Proof. Bouqet et al. prove that $2K_2$ free graphs (which have diameter 2 line graphs) have maximal stable sets that can be enumerated in polynomial time. Then, as matchings in the original graph correspond to a dominating set in the dual, the result follows. □

Theorem 5.7 (Hsu and Tsai)**.** *Constructing a dominating set for a proper circular arc graph can be done in polynomial time.*

Proof. First, cut at an arbitrary point to make the circular arc graph an interval graph. Then, use a sweep algorithm to process arcs, and record all intersections. Then, run a greedy algorithm that picks an interval covering the leftmost arc that extends the furthest to the right. □

Theorem 5.8 (Bouqet et al.)**.** *Finding a dominating set in claw-free diameter two graphs is solvable in polynomial time.*

Proof. Let G be a claw-free diameter 2 graph. Then, assume the size of the dominating set is larger than 4, as else we can brute force it in polynomial time. It follows that G has no $W-$join, and since the dominating set size is less than or equal to the maximal independent set size, the size of the maximal independent set is at least 3. We can also assume that $|V| > 13$, as else we can find a dominating set in constant time.

Next, assume that there is a pair of adjacent vertices u, v such that the neighborhood of u without v is a subset of the neighborhood of v without u. Then, for ever dominating set D of $G-u$, we have $N(u) \cap D \neq \emptyset$, so a dominating set of $G-u$ also dominates G. Also, removing u does not increase the diameter. Thus, we can search for all such pair u, v and remove them from G in polynomial time.

Thus, we can apply 5.5. If G is a line graph (which can be checked in polynomial time) we use 5.6, else we use 5.7. □

Definition 21. *A* Vertex Cover *of a graph is a subset of vertices such that each edge has at least one endpoint in this subset. This problem is known to be $NP-$complete.*

Theorem 5.9 (Bouqet et al)**.** *Finding a dominating set is NP-Complete for $K_{1,4}$ free graphs with diameter 2.*

Proof. This follows from a polynomial time reduction from vertex cover. From $I = (G = (V, E), k)$ an instance of Vertex Cover, we build an instance $I' = (G', \gamma)$ where G' is $K_{1,4}$-free with diameter 2 and $\gamma = k$.

We start by constructing $G' = (V', E')$. The vertices of V' are partitioned into V_1, E_1, E_2, S, s. We define these sets and the edges of G' as follows:

- For each vertex $v \in V$, there is a vertex $v_1 \in V_1$, that is, $V_1 = v_1 \mid v \in V$;
- For each edge $uv \in E$, there is a vertex $e^1_{uv} \in E_1$ and $e^2_{uv} \in E_2$, that is, $E_1 = e^1_{uv} \mid uv \in E$ and $E_2 = e^2_{uv} \mid uv \in E$;
- For each $u_1 \in V_1$, the vertices $u_1 \cup e^1_{uv} \mid u = u_1$ and $u_1 \cup e^2_{uv} \mid u = u_1$ form two cliques;
- $V_1 \cup S \cup s$ is a clique;
- For each pair $e, e' \in E_1 \cup E_2$ such that $N(e) \cap N(e') \cap V_1 = \emptyset$, there is a vertex $s_{e,e'} \in S$ and the two edges $s_{e,e'}e, s_{e,e'}e' \in E'$. Note that these edges e, e' correspond to copies of non-incident edges in G.

Since $V_1 \cup S \cup s$ is a clique and that every pair of vertices $e, e' \in E_1 \cup E_2$ has a common neighbor in $V_1 \cup S$, it follows that $\text{diam}(G') = 2$. We show that G' is $K_{1,4}$-free. For each vertex of G', we give a partition of its neighborhood into at most three cliques. For $u_1 \in V_1$: $N(u_1) \cap E_1$, $N(u_1) \cap E_2$, and $N(u_1) \cap (S \cup V_1 \cup s)$. For $e^i_{uv} \in E_i$, $i \in 1, 2$: $N(e^i_{uv}) \cap N(u_1)$, $N(e^i_{uv}) \cap N(v_1)$, and $N(e_{uv}) \cap S$. For $s_{e,e'} \in S$: $s_{e,e'}, e$, $s_{e,e'}, e'$, and $V_1 \cup S \cup s$. For the vertex s: $N(s) = S \cup V_1$. Therefore G' is $K_{1,4}$-free.

Let C, $|C| \leq k = \gamma$, be a vertex cover of G. Then its copy in V_1 is a dominating set of G' of size at most γ.

Let $I' = (G', \gamma)$ be a positive instance, so there exists Γ, $|\Gamma| \leq \gamma$ a dominating set of G'. From Γ we will construct a dominating set Γ' such that $\Gamma' \subseteq V_1$. Since $N(s) = V_1 \cup S$ we can assume that $s \notin \Gamma$. Let S_i be the vertices of S with two neighbors in E_i, that is, $S_i = s_{e,e'} \mid e, e' \in E_i$, $i = 1, 2$. Let $\Gamma_i = \Gamma \cap (E_i \cup S_i)$. Without loss of generality $|\Gamma_1| \leq |\Gamma_2|$. Let $\Gamma' = \Gamma \setminus \Gamma_2$. For each $e^1 \in \Gamma_1$, we add e^2 to Γ', and for each $s_{e^1,e'^1} \in \Gamma_1$, we add s_{e^2,e'^2} to Γ'. Since $G'[E_1 \cup S_1 \cup V_1]$ is isomorphic to $G'[E_2 \cup S_2 \cup V_1]$, it follows that Γ' is a dominating set of G' such that $|\Gamma'| \leq \gamma$.

Let E^0_i be the vertices $e^i_{uv} \in E_i$ such that $\Gamma' \cap N(e^i_{uv}) \cap V_1 = \emptyset$, $i = 1, 2$. For each $e^1_{uv} \in E^0_1$, there is $e^2_{uv} \in E^0_2$, and vice versa, because $N(e^1_{uv}) \cap V_1 = N(e^2_{uv}) \cap V_1$. Since $N(e^1_{uv}) \cap N(e^2_{uv}) \cap S = s_{e,e'}$, with $e = e^1_{uv}, e' = e^2_{uv}$, and that each vertex of S has exactly two neighbors in $E_1 \cup E_2$, it follows that $|E^0_1| \leq |S \cap \Gamma'|$. Then we remove the vertices of S from Γ' and we replace them by $u_1 \in V_1$ for each $e^1_{uv} \in E^0_1$. It follows that Γ' is a dominating set of G' such that $|\Gamma'| \leq \gamma$. Note that $\Gamma' \subseteq V_1$.

Let C be the copies of the vertices of $\Gamma' \cap V_1$ in G. Since each vertex $e^1_{uv} \in E_1$ has a neighbor in $\Gamma' \cap V_1$, it follows that C is a vertex cover of G such that $|C| \leq k$. □

Definition 22. *A split graph S is a graph whose vertices can be partitioned into $S = (K, I)$, where K is a clique and I is an independent set.*

Theorem 5.10. *Given a split graph, we can find the split in polynomial time.*

Proof. First, we sort all vertices by degree. Find the largest k for which we have k vertices of degree $k-1$ or larger. Consider all vertices with degree exactly $k-1$. Then, these vertices must be connected to all vertices in the k clique, and we can pick any to be in the clique. □

Definition 23. *A vertex v is called* simplicial *when $N(v)$ is a clique.*

Lemma 5.11 (Bouqet et al.)**.** *If u, v are vertices of a graph such that $N(u) \subset N(v)$, and v is simplicial, then the dominating set of G is not affected by the removal of v.*

Proof. For any graph containing a simplicial vertex, there is a dominating set that does not contain this vertex. Let S be such a set that does not contain v. Then S is a dominating set of $G - v$. Then, it follows that u is simplicial, and that $uv \notin E$. Thus, there is a set $|S'|$ that dominates and does not contain u, with $|S| = |S'|$. □

Theorem 5.12 (Bouqet et al)**.** *Finding a dominating set is NP-complete for triangle-free graphs with diameter 2*

Proof. We give a polynomial transformation from Dominating Set, which is NP-complete for split graphs with diameter 2 (see [9]). From $I = (G, k)$ an instance of Dominating Set, we build an instance $I' = (G', k')$.

In $I = (G, k)$, $G = (K \cup S, E)$ is a split graph with diameter 2 where K is a clique and S is a stable set. Let $u, v \in S$. First, since the vertices of S are simplicial, it follows from 5.11 that we can suppose that $N(u)N(v)$ and $N(v)N(u)$. Second, since $\text{diam}(G) = 2$, there exists $w \in K$ such that $u - w - v$ is a path in G.

From G we build $G' = (V', E')$ as follows. We take a copy K_1 of K and two copies S_1, S_2 of S. For the sake of simplicity, for $v \in K$, its copy in K_1 is denoted by v_1, whereas for $v \in S$, its copies in S_1, S_2 are denoted by v_1, v_2, respectively. We then add two vertices t and s. For each pair $u \in K$, $v \in S$, if $uv \in E$, then we add the edge u_1v_1; otherwise, we add the edge u_1v_2. For every $v \in S$, we add the edge v_1v_2. Then we make t complete to K_1 and s complete to S_2. Last, we add the edge st. Note that t, K_1, s, S_1, S_2 is a partition of G' into stable sets. Finally, we take $k' = k + 1$.

We show that G' is triangle-free. Since $N(t) = K_1 \cup s$ and $N(s) = S_2 \cup t$ are two stable sets, it follows that t and s cannot be in a triangle. Thus, if a triangle exists, it has one vertex $u_1 \in K_1$, one vertex in S_1, and one vertex in

S_2. So this triangle contains the edge v_1v_2. But when u_1v_1 is an edge, u_1v_2 is not an edge, and vice versa. So G' is triangle-free.

We show that $\text{diam}(G') = 2$. We observe that t and s are at distance at most two from any vertex of the graph. So we can focus on the vertices of $K_1 \cup S_1 \cup S_2$. Since t is complete to K_1 and s is complete to S_2, for any pair $v_1, u_1 \in K_1$ (respectively $v_2, u_2 \in S_2$), there exists the path $v_1 - t - u_1$ (respectively $v_2 - s - u_2$).

Since $\text{diam}(G) = 2$, for any pair $v_1, u_1 \in S_1$, there exists $w_1 \in K_1$ such that $v_1 - w_1 - u_1$ is a path of G'. Now let $u_1 \in S_1$, $v_1 \in K_1$ (respectively $u_2 \in S_2$, $v_1 \in K_1$), such that $uv \notin E$ (respectively $uv \in E$). Then $u_2v_1 \in E'$ (respectively $u_1v_1 \in E'$), so $u_1 - u_2 - v_1$ (respectively $u_2 - u_1 - v_1$) is a path of G'. Now let $u_1 \in S_1$, $v_2 \in S_2$, $u \neq v$. From Lemma 4.1, we can assume that there exists $w \in N(u)$, $w \notin N(v)$. Therefore $u_1w_1, v_2w_1 \in E'$, and $u_1 - w_1 - v_2$ is a path in G'. So $\text{diam}(G') = 2$.

Let D be a dominating set of G with $|D| \leq k$. Let D' be the set of the copies of the vertices of D in $K_1 \cup S_1$. Then $D' \cup t$ is a dominating set of G' and $|D' \cup t| \leq k + 1 = k'$.

Conversely, let D' be a dominating set of G' with $|D'| \leq k' = k + 1$. Since $N(s) = S_2 \cup t$, it follows that $|D' \cap (t, s \cup S_2)| \geq 1$.

First, suppose $S_2 \cap D' = \emptyset$. So $|D' \cap t, s| \geq 1$. For each $v_1 \in D' \cap S_1$, if any, let a unique $u_1 \in N(v_1) \cap K_1$. Then let

Second, if $|S_2 \cap D'| \geq 1$ and $t \in D'$, then for each $v_2 \in D' \cap S_2$, let a unique $u_1 \in N(v_1) \cap K_1$ (where v_1 is the neighbor of v_2 in S_1). Then let

Third, if $|S_2 \cap D'| \geq 1$ and $t \notin D'$, then for each $v_1 \in S_1$ which is not dominated by a vertex of K_1 or by itself, we have that v_1 is dominated by v_2, its neighbor in S_2. Let any $w \in N(v_1) \cap K_1$. Since $t \notin D'$, we have that w is dominated either by $u_1 \in N(w) \cap S_1$, $u_1 \neq v_1$, or by $u_2 \in S_2$, $u_2 \neq v_2$. In the first case, we replace u_1 by w in D', in the second case we replace u_2 by w in D'. Then we take $\bar{D} = D' \cap (K_1 \cup S_1)$.

In all cases, we take D to be the copies of the vertices of $\bar{D}$ in G. We have that D is a dominating set of G with $|D| \leq k$. $\square$

6 Family of all Diameter 2 Graphs

6.1 Well Quasi Ordering

Definition 24. *A* well-quasi-order *is a preorder $(P, \leq)$, such that for any infinite sequence $\{x_i\}_{i \in I}$, there exists some $i < j$ with $x_i \leq x_j$.*

Alternatively, this is expressed by satisfying two conditions. First, there must be no infinite antichains (set in which no two elements are comparable), so for any infinite sequence in P, there is some pair of elements that are comparable. Second, there must be no strictly decreasing infinite sequence in P.

Definition 25. *A* cograph *is a graph which does not contain P_4 as an induced subgraph.*

Theorem 6.1 (Damaschke)**.** *Cographs are well-quasi ordered under the subgraph relation.*

Theorem 6.2. *The family of claw-free and C_5 free diameter 2 graphs are well-quasi ordered.*

Proof. If a diameter 2 graph has an induced path $P_4 = v_1v_2v_3v_4$, then we know there must be edges $v_1v_5, v_4v_5 \in E$ in order for $d(v_1, v_4) \leq 2$ to hold. Then, we could still have this graph not be a cograph if v_2v_5 or $v_3v_5 \in E$, as either would imply that $v_1v_2v_3v_4v_5v_1$ is not an induced 5 cycle. Forbidding claws and 5 cycles thus ensures that we cannot have P_4 as a subgraph. Then, all remaining graphs are cographs, and so the remaining family is a collection of cographs, and thus is well-quasi ordered. □

6.2 Counting Graphs

Definition 26. *We call a graph G* strongly regular *of degree k if it can be described as (n, d, p, q), where every pair of adjacent neighbors have p neighbors in common, and every nonadjacent pair has q neighbors in common. Such a graph is* primitive *if both itself and its complement are connected. Note that these graphs are diameter 2 (and distance regular) if $q > 0$.*

Lemma 6.3 (Haemers)**.** *Let f_n be the multiplicity of λ_n. Then,*

$$\gamma(G) \geq \max\{1 + f_n, 1 - \frac{\lambda_n}{\lambda_2}\} \tag{23}$$

Proof. Let $\gamma \leq f_n$. Then, $\lambda_n = \lambda_{n-\gamma+1}$. By observing that

$$(\gamma - 1)\lambda_{k+1} \geq -\lambda_{n-k(\gamma-1)} \tag{24}$$

the result follows, with $k = 1$. □

Lemma 6.4 (Haemers)**.** *If G is primitive and strongly regular, and not the pentagon or the complete $\gamma-$bipartite graph, then*

1. $d \leq -\lambda_n(\gamma(G) - 1)$
2. $-\lambda)n \leq \lambda_2(\gamma(G) - 1)$
3. $\lambda_2 \leq \gamma(G) - 1$

Proof. First, we prove that $\gamma(G) \geq \max\{1 - \frac{\lambda_1}{\lambda_n}, 1 - \frac{\lambda_n}{\lambda_2}\}$. If $n \leq 28$, we verify this by computer checking. If $\lambda_2 < 2$, then either it is equal to 1, or G is the conference graph (and thus n<25). Else, strongly regular graphs with $\lambda_2 = 1$ were proven by Seidel [CITE] to satisfy $n \leq 28$, be a ladder, complement of a lattice, of complement of a triangular graph, all of which satisfy the bound on $\gamma(G)$. Finally, let $\lambda_2 \geq 2$. If G is imprimitive, the result follows, so assume G

is primitive. Then, if the result did not hold, we have $\lambda_1 < n$, and $f_n\lambda_n + (n - 1 - f_n\lambda_2 + \lambda_1 = 0$ implies

$$f_n^2 < -\frac{f_n\lambda_n}{\lambda)2} \tag{25}$$

$$= n - 1 - f_n + \frac{\lambda_1}{\lambda_2} \tag{26}$$

$$< \frac{3}{2}n - f_n \tag{27}$$

So $f_n^2 + 3f_n < \frac{3}{2}n + 2\sqrt{\frac{3}{2}n}$, thus $n < 24$, we have a contradiction, and our bound on $\gamma(G)$ follows.

Then, we can deduce (1) and (2).

Since G is primitive, $0 < q = d - \lambda_2\lambda_n$, so by (1) we have $\gamma(G) - 1 \geq -\frac{d}{\lambda_n} > \lambda_2$ □

Lemma 6.5 (Vieta)**.** *For a strongly regular graph G, we have* $q - d = \lambda_2\lambda_n$

Theorem 6.6 (Haemers)**.** *For any* $n \in \mathbb{N}$*, the number of primitive strongly regular graphs with chromatic number n is finite.*

Proof. If G is primitive, then $q \geq 1$, and so by 6.4

$$n \leq nq = (d - \lambda_2)(d - \lambda_n) \leq d(d - \lambda_n) \leq d(d - \lambda_n) < \gamma(\gamma - 1)^5 \tag{28}$$

□

Theorem 6.7 (Hoffman and Singleton)**.** *Every graph with diameter 2 and girth 5 is a moore graph that is k-regular with* $k^2 + 1$ *vertices, with* $k \in \{2, 3, 7, 57\}$*, with the existence of* $k = 57$ *being a mystery.*

6.3 Mycielskian

Definition 27. *The* Mycielskian *M(G) of a graph G is a construction obtained by adding n+1 vertices such that, if the original vertices are* $v_1, \ldots, v_n$*, we have for each edge* v_iv_j *new edges* u_iv_j *and* u_jv_i*. Call* $u_1 \ldots, u_n$ *auxiliary vertices. Finally, connect the n+1th new vertex to all auxiliary vertices.*

Theorem 6.8. *If G is diameter 2, then M(G) is as well.*

Proof. Consider all vertices in $G \leq M(G)$. Then, these can reach all vertices in G in 2 steps by assumption, and all new auxillary vertices by either directly traveling to it or by traveling through a common vertex. Furthermore, they can reach the final new vertex by traveling through any auxiliary vertices. The new vertices form a star, so they can all reach each other as well. □

Theorem 6.9. *The Mycielskian construction increases the chromatic number.*

Proof. Color $G \leq M(G)$ as you normally would. Then, we must use all of the colors in the auxiliary vertices, and the final new vertex increases the chromatic number by 1. □

Theorem 6.10 (Mycielski)**.** *The Mycielskian constrution does not add cliques*

Proof. The only new triangles must be of the form $v_i v_j u_k$, where $v_i v_j v_k$ is a triangle in G. □

Lemma 6.11. *If $G \subset G'$, then $M(G) \subset M(G')$*

Proof. This follows from the construction of $M(G)$ □

6.4 C_i and P_j freeness

This is open for P_t free graphs when $t \geq 8$. For $t \geq 2$, we have that P_t free graphs are a subclass of $C_{<t}$ free graphs (both due to Matin et al). Rojas and Stein prove poly time for $(C^{odd}_{<t-3}, P_t)$ free. It is also polynomial time solvable for the following diameter 2 graphs:

- Diamond-free graphs with an articulation neighborhood but without nested neighborhoods
- (C_3, C_4)-free graphs
- $K_{2,1,r}$-free graphs for every $r \geq 1$;
- $S_{1,2,2}$-free graphs

[By Martin et al]

Martin et al expands to C_5 free, C_6 free, and (C_4, C_t) free for $t \in \{3, 5, 6, 7, 8, 9\}$. We have poylnomial time 3 coloring for $(C_4,\ C_t)$ free when $t \geq 10$, solved by Klimosova and Sahlot. Finally, (C_3, C_s) with $s \geq 8$ remains open.

Let G be a (C_3, C_8) free graph. Then, we can assume there is an induced 5-cycle, as otherwise we would have a (C_3, C_5) free graph, which is polynomial time colorable. Call our cycle C_5 on vertices 1,2,3,4,5. Let $N_1 = N(C_5)$, and let $N_2 = N(N_1) \backslash C_5$. Let Col_1 be the list of all vertices in N_1 with list size 1, and similarly Col_2 the list of all vertices in N_2 with list size 1.
Without a loss of generality, color 1 and 3 a, 2 and 4 b, and 5 c. Let A be the set of all vertices in N_1 connected to 1 or 3 but not in Col_1, let B be the set of all vertices in N_1 connected to 2 or 4 but not in Col_1, and let C be the set of all vertices connected to 5 but not in Col_1.
Since the cycle is colored, all vertices in $N(C_5)$ have list size at most 2, so our vertices in sets A, B, C have list size exactly 2. As a result of this, vertices in A are not connected to 2,4,5, similar for B and 1,3,5, and C for 1,2,3,4 (as otherwise they would have list size 1).
Further partition A, B, C so that A_1 are the vertices connected to 1 but not 3, A_3 are the vertices connected to 3 but not 1, and A_{13} are connected to both. Similarly construct B_2, B_4, B_{24}. Then, partition N_2 into L_3, L_2, and L_1, collections of vertices with list 3, list 2, and list 1.

Lemma 6.12. *If $v \in L_3$, then v has an edge in A, B, C*

This result follows from G being diameter 2. WLOG let v not have an edge in C. Then, v cannot reach 5 in 2 steps.

We must have a 7 cycle. Then, we can assume two cases for the coloring of this cycle. From this, we can deduce rules about connections between groups of vertices, which informs coloring.

6.5 Counting Forbidden Subgraphs

Theorem 6.13 (Pan, Stefankovic)**.** *There is an infinite family of forbidden diameter two subgraphs that force 4 or higher chromatic number*

Proof. Consider a diameter 2 graph G with 3^k vertices. Let $V = \{\{0, 1, 2\}^k\}$,, and let our edge set satisfy $E = \{\{a, b\} | \forall i a_i \neq b_i\}$. This graph is 3 colorable and diameter 2 with $3k$ colorings. To see that G satisfies the diameter 2 constraint, consider two vertices without an edge u, v. Then, they differ by at least one coordinate, however there is at least one third node w that differs from all coordinates from both nodes, so there is an edge between u, w and w, v. To find a coloring, partition all vertices by their ith coordinate, then each set will be independent. Next we generate forbidden subgraphs. We add k edges between vertices sharing a_i, b_i, and the ith edge removes the ith coloring. Each edge addition creates a graph that is not a subgraph of prior graphs, and we can do this for any k, creating an arbitrarily large family. □