# Honors Thesis - Iterated Monodromy Groups

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**Abstract**

Do this last

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# 1 Introduction

Iterated rational functions, iterated galois groups, analogue to tate module?, embedding into tree automorphisms, exact sequence, constant field extension, previous work

# 2 Preliminaries

Still need to do $\pi_{x,i}$

## 2.1 Tree automorphisms and wreath products

**Definition 2.1.** *Let $G$ and $H$ be groups, and let $G$ act on a set $\Omega$ on the left. Their* **wreath product** $H \wr_\Omega G$ *with respect to this action is the semidirect product $H^\Omega \rtimes G$, where the (right) action of $g \in G$ on $\eta \in H^\Omega$ is given by*

$$\eta^g := (\eta_{g\omega})_{\omega \in \Omega}.$$

For the purposes of this paper, we will always have $\Omega = \{1, \dots, d\}$ for some $d \geq 1$, and $G$ will always be a subgroup of $S_d$ acting on $\Omega$ in the usual way. In this case, we will omit $\Omega$ if this does not result in confusion. Additionally, we will usually denote by $\sigma \in S_d$ the $d$-cycle $(1\ 2\ \dots\ d)$.

**Definition 2.2.** *Let $d \geq 1$ be a natural number. The* **iterated wreath products** *of $G \leq S_d$ are the groups $W_n(G) = G^{\wr n}$ defined by setting $G^{\wr 0} := \{1\}$ and*

$$G^{\wr(n+1)} := G^{\wr n} \wr G.$$

These groups act naturally on $d$-branching trees, which we define next.

**Definition 2.3.** *Let $d \geq 1$ and $n \geq 0$ be integers, and let $X = \{1, \dots, d\}$. Consider the set $V := \bigsqcup_{0 \leq i \leq n} X^i$ of left-expanding words of length at most $n$ on $X$ (where $X^0 := \{\epsilon\}$ consists of the empty word $\epsilon$). The* **height $n$, $d$-branching rooted tree** *$T_d^n$ is the graph on $V$ with an edge between $x_k \dots x_1$ and $x_{k+1} \dots x_1$, for all $k \geq 0$ and $x_1, \dots, x_{k+1} \in X$.*

Given $G \leq S_d$, the action of $G^{\wr n}$ on $T_d^n$ is given recursively as follows. The group $G^{\wr 0} = \{1\}$ acts trivially on $T_d^0 = \{\epsilon\}$, and $G^{\wr 1} \cong G$ acts on $T_d^1 = X \sqcup \{\epsilon\}$ by permuting the elements of $X$. For $n \geq 1$, the action of an element $w = (w_1, \dots, w_d)g \in G^{\wr(n+1)}$ on a word $x_k \dots x_1 \in T_d^{n+1}$ is given by

$$(x_k \dots x_1)^w := (x_k \dots x_2)^{w_{x_1}} x_1^g.$$

From this action, we get embeddings $G^{\wr n} \hookrightarrow \mathrm{Aut}(T_d^n)$. In fact, it is straightforward to see that $\mathrm{Aut}(T_d^n) \cong S_d^{\wr n}$. We also get isomorphisms $G^{\wr(n+1)} \cong G \wr G^{\wr n}$, and hence projection maps $G^{\wr(n+1)} \twoheadrightarrow G^{\wr n}$, which can be thought of as restricting the action of $G^{\wr(n+1)}$ to the first $n$ levels of the tree.

**Definition 2.4.** *Let $d \geq 1$ be a natural number. The* **infinite iterated wreath product** *of $G \leq S_d$ is the group $W(G) := G^{\wr\infty}$ defined by*

$$W(G) := \varprojlim W_n(G),$$

*where the projections in the inverse system are the maps $G^{\wr(n+1)} \twoheadrightarrow G^{\wr n}$ from above. For $n \geq 0$, we denote the projection $W(G) \twoheadrightarrow W_n(G)$ by $p_n$.*

The $d$-branching trees form an inverse system of their own, where the projection $T_d^{n+1} \twoheadrightarrow T_d^n$ fixes the first $n$ levels and sends each node at the $(n+1)$-th level to its parent node at the $n$-th level.

**Definition 2.5.** *The* **infinite, $d$-branching rooted tree** *$T_d^\infty$ is the limit of the inverse system above:*

$$T_d^\infty := \varprojlim T_d^n.$$

An element of $T_d^\infty$ is an infinite, left-expanding word on $X = \{1, \dots, d\}$. Since $(G^{\wr n})_{n \geq 0}$ and $(T_d^n)_{n \geq 0}$ are families of finite structures, both $W(G)$ and $T_d^\infty$ are naturally endowed with profinite topologies. One can define an ultrametric on each of $W(G)$ and $T_d^\infty$ that agrees with the profinite topology. Namely, for $w_1, w_2 \in W(G)$, we let

$$d(w_1, w_2) := 2^{-N},$$

where $N \coloneqq \inf\{k \geq 1 \colon p_k(w_1) \neq p_k(w_2)\}$ is the first level at which $w_1$ and $w_2$ differ. (Here, we take $2^{-\infty}$ to be zero.) Similarly, for $x = (\dots x_2 x_1), y = (\dots y_2 y_1) \in T_d^\infty$, we let

$$d(x, y) \coloneqq 2^{-M},$$

where $M \coloneqq \inf\{k \geq 1 \colon x_k \neq y_k\}$.

Furthermore, $W(G)$ has a natural action on $T_d^\infty$ given by

$$\Big((\dots x_2 x_1)^w\Big)_r \coloneqq \Big((x_r \dots x_1)^{p_r(w)}\Big)_r.$$

In other words, the result of applying $w \in W(G)$ to an infinite word $x = (\dots x_2 x_1) \in T_d^\infty$ can be computed up to the $r$-th level by projecting (restricting) $w$ to $G^{\wr r}$ and applying it to the first $r$ levels of $x$. That the action is well defined follows from the properties of inverse limits. It is continuous with respect to the profinite topologies, and it yields an embedding $W(G) \hookrightarrow \operatorname{Aut}(T^\infty) \cong S_d^{\wr\infty}$.

In light of this action, we associate with each subgroup $H \leq W(G)$ its *level n stabilizer* $\operatorname{Stab}_n H \coloneqq H \cap \ker p_n$, i.e., the subgroup that fixes the first $n$ levels of the tree. Then $\{\operatorname{Stab}_n W(G)\}_{n\geq 1} = \{\ker p_n\}_{n \geq 1}$ is a basis of open balls at the identity element in $W(G)$.

Noting that

$$T_d^\infty = \bigsqcup_{x \in X} T_d^\infty x \cong \bigsqcup_{1 \leq i \leq d} T_d^\infty,$$

one can see that $W(G) = (\operatorname{Stab}_1 W(G)) \rtimes G \cong W(G) \wr G$. Intuitively, this means that an element $w \in W(G)$ can be written as $(w|_1, \dots, w|_d)g$, where $g$ is the action of $w$ on the first level in $T_d^\infty$, and $w|_1, \dots, w|_d \in W(G)$ encode the action of $w$ on each of the $d$ copies of $T_d^\infty$ with roots (prefixes) in $X$. We thus obtain projections

$$\pi_j \colon \operatorname{Stab}_1 W(G) \twoheadrightarrow W(G); w \mapsto w|_j,$$

for $1 \leq j \leq d$. This provides a compact way of representing the elements of $W(G)$ which have a recursive action on $T_d^\infty$.

*Example.* Let $\sigma = (1\ 2\ \dots\ d) \in S_d$. The *standard odometer* $\omega \in W(S_d)$ is the element defined by

$$\omega = (1, \dots, 1, \omega)\sigma.$$

Its action on $T_d^\infty$ can be computed by repeatedly expanding the equation above:

$$\omega = (1, (1, \omega)\sigma)\sigma = (1, (1, (1, \dots)\sigma)\sigma)\sigma.$$

We will address the issues of existence and uniqueness of such elements in the next subsection. It turns out that, under certain conditions, one can use systems of mutual recursion to define several elements at once, or even recursive systems of conjugacies to define conjugacy classes. But before we turn to that, we introduce a bit more notation that will streamline inductive arguments in later sections:

**Definition 2.6.** *We say two elements $w_1, w_2 \in W(G)$ are* **equal up to level** *$n$, and write $w_1 =_n w_2$, if $w_1 \equiv w_2 \pmod{\operatorname{Stab}_n W(G)}$. We say they are* **conjugate up to level** *$n$ if $p_n(w_1) \sim p_n(w_2)$ in $W_n(G)$.*

**Definition 2.7.** *For a subgroup $H \leq W(G)$, and elements $w_1, w_2 \in H$, we say $w_1$ and $w_2$ are* **conjugate in** *$H$, and write $w_1 \sim_H w_2$*

## 2.2 Systems of recursion

In what follows, let $G \leq S_d$ be a group, $W_n := W_n(G)$ be its iterated wreath products, and $W := W(G)$ be its infinite iterated wreath product. For a group $H$, we let $\mathcal{C}(H) := \{[h] : h \in H\}$ be the set of its conjugacy classes. For brevity, we will denote $\mathcal{C}(W)$ by just $\mathcal{C}$.

For a space $X$ and a function $F = (F_1, \dots, F_n) : X^n \to X^n$, one can associate to $F$ a *system of equations*

$$x_i = F_i(x_1, \dots, x_n)$$

in $n$ variables $x_1, \dots, x_n \in X$ (where $i$ above ranges from 1 to $n$). A *solution* to this system is a tuple $x = (x_1, \dots, x_n) \in X^n$ that satisfies $F(x) = x$, i.e., a fixed point of $F$. Hence, if $X$ is a compact metrizable space and $F$ is a continuous, contracting map, the Banach fixed-point theorem implies the existence of a unique solution $x \in X^n$ to the system above, which can be obtained as the limit $\lim_{N\to\infty} F^N(v)$, for any $v \in X^n$. (Here, $X^n$ is taken with the product topology.)

Since $W$ is a compact metrizable space under the profinite topology, the above reasoning applies straightforwardly. Let us record this result:

**Definition 2.8.** *A* **system of recursions** *in $W$ is the system of equations associated to a function $F : W^n \to W^n$. We call it* **continuous** *(resp.* **contracting***) if $F$ is continuous (resp. contracting).*

**Lemma 2.9.** *A continuous, contracting system of recursions in $W$ has a unique solution.*

We would like to carry over the argument above to $\mathcal{C}$. We must then show that $\mathcal{C}$ can be endowed with a topology under which it is a compact metrizable space. The following lemma will help us define an ultrametric on $\mathcal{C}$ analogous to that on $W$. The end-goal here is to show that the natural quotient topology of $\mathcal{C}$ agrees with the one obtained from the ultrametric and makes $\mathcal{C}$ a profinite space.

**Lemma 2.10.** *Let $x \in W$ be arbitrary. Then $[x] = \{x^w : w \in W\}$ is a closed subset of $W$.*

*Proof.* Consider the map $\varphi_x : W \to W$ given by $\varphi_x(w) = x^w$. Then $\varphi$ is a continuous map. Since $W$ is a profinite group, it is compact and Hausdorff. Hence $[x] = \varphi_x(W)$ is a compact subset of the Hausdorff space $W$, and is therefore closed. ☐

**Corollary 2.11.** *If $x, y \in W$ satisfy $x \sim_n y$ for arbitrarily large (and therefore all) natural numbers $n$, then $x \sim_W y$.*

*Proof.* For $n \geq 1$, take a $g_n \in W$ such that $x^{g_n} =_n y$. Then the sequence $(x^{g_n})_{n\geq 1}$ converges in $W$ to $y$. Thus, $y \in \overline{[x]} = [x]$. ☐

We are now ready to construct the desired metric. For $x, y \in W$, we define

$$d([x], [y]) := 2^{-N},$$

where $N := \inf\{k \geq 1 : x \not\sim_k y\}$. Again, we take $2^{-\infty}$ to equal 0. It is clear that $N$ does not change if we replace $x, y$ by $W$-conjugates of them. Therefore, $d([x], [y])$ is well-defined. Corollary 2.11 shows that $d([x], [y]) = 0 \iff x \sim_W y$. Symmetry and the ultrametric inequality are straightforward to check. Thus, $d$ is an ultrametric on $\mathcal{C}$.

**Lemma 2.12.** *The ultrametric topology on $\mathcal{C}$ agrees with the quotient topology.*

*Proof.* Let $q\colon W \twoheadrightarrow \mathcal{C}$ be the quotient map, and denote by $B_c(\gamma, r) \subseteq \mathcal{C}$ the open ball with center $\gamma$ and radius $r$ under the ultrametric. For all $x \in W$ and $k \geq 0$, we have

$$B_c\big([x], 2^{-k-1}\big) = \big\{[y]\colon y \in W, y \sim_k x\big\} = \{[y]\colon y \in W, y =_k x\} = q(x \operatorname{Stab}_k W).$$

Note that $\big\{B_c\big([x], 2^{-k-1}\big)\big\}_{k\geq 0}$ is a basis of open balls at $[x]$ in the metric topology. Hence, to conclude, it suffices to show that $\{q(x \operatorname{Stab}_k W)\}_{k\geq 0}$ is a basis at $[x]$ in the quotient topology.

First, note that for all $x \in W$ and $k \geq 0$,

$$q^{-1}\big(q(x \operatorname{Stab}_k W)\big) = \bigcup_{w\in W} \big(x \operatorname{Stab}_k W\big)^w = \bigcup_{w \in W} x^w \operatorname{Stab}_k W,$$

which is a union of open subsets in $W$. Hence, $q(x \operatorname{Stab}_k W)$ is open in the quotient topology. Now, take an arbitrary neighborhood $U$ of $[x]$ in the quotient topology. Then $x \in q^{-1}(U) \subseteq W$. Hence, there exists an $l \geq 0$ such that the basis set $x \operatorname{Stab}_l W$ is contained in $q^{-1}(U)$. Therefore, $q(x \operatorname{Stab}_l W) \subseteq q(q^{-1}(U)) = U$. This shows that $\{q(x \operatorname{Stab}_k W)\colon x \in W, k \geq 0\}$ is a basis in the quotient topology, concluding the proof. □

The final step, before returning to systems of recursions, is to show that $\mathcal{C}$ is a profinite space under the quotient topology. This will imply that $\mathcal{C}$ is a compact Hausdorff metric space, providing all the necessary conditions to carry the contracting function argument for systems of conjugacy recursions, to be defined later.

For $0 \leq m \leq n$, we define a map $\pi^n_m\colon \mathcal{C}(W_n) \to \mathcal{C}(W_m)$ by the rule $[p_n(x)]_{W_n} \mapsto [p_m(x)]_{W_m}$, for all $x \in W$. Once can check that these maps are well-defined and surjective, and that $\{\mathcal{C}(W_n)\}_{n\geq 0}$ is an inverse system under them.

**Lemma 2.13.** *Take $\mathcal{C}$ with the quotient topology. Then*

$$\mathcal{C} \cong \varprojlim \mathcal{C}(W_n).$$

*Proof.* One can see that the maps

$$p^c_n\colon \mathcal{C} \to \mathcal{C}(W_n); [x] \mapsto [p_n(x)]$$

are well-defined and surjective for all $n \geq 0$, and that

$$\big(p^c_n\big)^{-1}\big([p_n(x)]\big) = \big\{[y]\colon y \in W, p_n(y) \sim_{W_n} p_n(x)\big\} = q(x \operatorname{Stab}_n W).$$

Hence, the maps $p^c_n$ are also continuous. It is also easy to check that $\pi^n_m \circ p^c_n = p^c_n$ for all $n \geq m \geq 0$, i.e., that the maps $p^c_n$ are compatible with those of the inverse system.

Now, suppose $Y$ is any space with continuous maps $h_n\colon Y \to \mathcal{C}(W_n)$ that are compatible with the inverse system maps. We show that there is a unique map $\psi\colon Y \to \mathcal{C}$ such that $h_n = p^c_n \circ \psi$ for all $n \geq 0$. To construct such a map, take an arbitrary $y \in Y$, and note that since the maps $p^c_n$ are surjective, there exists, for each $n \geq 0$, an $x_n \in W$ such that $p^c_n([x_n]) = h_n(y)$. Now, for all $n \geq m \geq 0$, we have

$$p^c_m([x_m]) = h_m(y) = \pi^n_m(h_n(y)) = \pi^n_m(p^c_n([x_n])),$$

i.e., $x_m \sim_m x_n$. Hence, we can inductively construct $g_0, g_1, \dots \in W$ such that $x^{g_{n+1}}_{n+1} =_n x^{g_n}_n$ holds for all $n \geq 0$. This implies that $x^{g_n}_n =_m x^{g_m}_m$ for all $n \geq m \geq 0$. Hence, the sequence $\big\{x^{g_n}_n\big\}_{n\geq 0}$

has a limit $x$ in $W$. We define $\psi(y) \coloneqq [x] \in \mathcal{C}$. We have thus obtained a function $\psi$ that satisfies $h_n = p_n^c \circ \psi$.

Now we show that this function is continuous. Consider an arbitrary basic open set $U = q(x\,\mathrm{Stab}_k(W))$ of $\mathcal{C}$, where $x \in W$ and $k \geq 0$. Then $V \coloneqq p_k^c(U) = \{[p_k(x)]\}$ is an open subset of $\mathcal{C}(W_k)$, and we have $(p_k^c)^{-1}(V) = U$. Hence,

$$\psi^{-1}(U) = \psi^{-1}((p_k^c)^{-1}(p_k^c(U))) = h_k^{-1}(V)$$

is an open subset of $Y$ by continuity of $h_k$. The preimages of the basic open sets under $\psi$ are hence open, which means that $\psi$ is continuous.

All is left is to show that $\psi$ is the unique map with the property that $h_n = p_n^c \circ \psi$ for all $n \geq 0$. Suppose $\psi'$ is another such map. Take an arbitrary $y \in Y$, and let $x, x' \in W$ be such that $[x] = \psi(y)$ and $[x'] = \psi'(y)$. Then, for all $n \geq 0$, we have

$$[p_n(x)] = p_n^c([x]) = h(y) = p_n^c([x']) = [p_n(x')].$$

Thus, $x \sim_n x'$ for all $n \geq 0$. Using Corollary 2.11, we deduce that $x \sim_W x'$, meaning

$$\psi(y) = [x] = [x'] = \psi'(y).$$

As $y \in Y$ was arbitrary, we have just shown that $\psi \equiv \psi'$. This concludes the proof, since it shows that $\mathcal{C}$ satisfies the definition of the limit of the inverse system $\{\mathcal{C}(W_n)\}_{n\geq 0}$. □

Now, we know that $\mathcal{C}$ is at once a profinite space, hence compact and Hausdorff; a metric space, which further allows us to apply the Banach fixed-point theorem; and a quotient space of $W$, which lets us easily construct continuous maps to and from $\mathcal{C}^n$. We record this crucial fact in the following proposition, before finally writing down the $\mathcal{C}$-analogues of systems of recursions.

**Proposition 2.14.** *The space $\mathcal{C}$ under the natural quotient topology is metrizable (with an ultrametric) and profinite.*

Given this, the conjugacy versions of Definition 2.8 and Lemma 2.9 are the following:

**Definition 2.15.** *A* **system of conjugacy recursions** *in $W$ is the system of equations associated to a function $F\colon \mathcal{C}^n \to \mathcal{C}^n$. We call it* **continuous** *(resp.* **contracting***) if $F$ is continuous (resp. contracting).*

**Lemma 2.16.** *A continuous, contracting system of conjugacy recursions in $W$ has a unique solution in $\mathcal{C}^n$.*

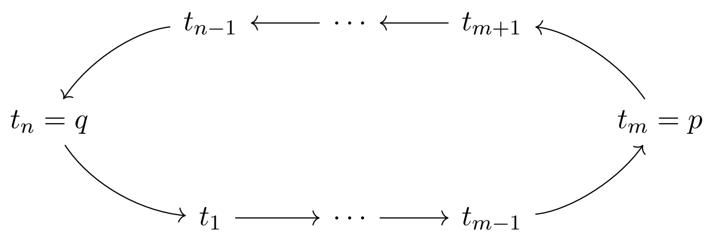
Lemma 2.9 will later allow us to uniquely specify a number of elements of $W$ simultaneously via mutual recursion, while Lemma 2.16 will allow us to specify a number of conjugacy classes in $W$ using function on $W^n$ that are invariant under conjugation.

### 2.3 Iterated monodromy groups

## 3 Cyclic Case

From now on, we will be concerned with the following special case. Let $f \in K(x)$ be a rational function of degree $d$, not divisible by the characteristic of $K$, with exactly 2 critical points, $p$ and $q$.

Assume further that these critical points are periodic under $f$ and belong to the same orbit. That is, there exist $n > m \geq 1$ and $t_1, \ldots, t_n \in \overline{K}$ such that $t_m = p$, $t_n = q$, and $f(t_i) = t_{i+1}$ for all $i \geq 1$, where we take the indices modulo $n$. The diagram below illustrates the $f$-orbit of $p$ and $q$.




We shall restrict our attention to the case where $n > 2$. This is because when $n = 2$, we have $f(p) = q$ and $f(q) = p$. By conjugating with a Möbius transformation, we may then assume $\{p, q\} = \{0, \infty\}$, which results in $f$ being a conjugate of $x^{-d}$. The iterated monodromy group of $x^{-d}$ is easy, right?

Let us then assume that $n > 2$. By swapping $p$ and $q$ if necessary, we can ensure that $m > 1$, i.e. $t_1$ is a non-critical branch point.

generators of IMG

### 3.1 Model generators and self-similarity

**Definition 3.1.** *A* **model group** $M$ *is a profinite subgroup of* $W$ *that is topologically generated by elements* $a_1, \ldots, a_n \in W$ *which satisfy the following recursions:*

$$a_i = \begin{cases} (a_n, 1, \ldots, 1)\sigma, & i = 1 \\ (1, \ldots, 1, a_{i-1}), & 1 < i \leq m \\ (1, \ldots, 1, a_m)\sigma^{-1}, & i = m+1 \\ (a_{i-1}, 1, \ldots, 1), & m+1 < i \leq n, \end{cases} \tag{1}$$

*as well as the product relation* $a_1 \ldots a_n = 1$.

Note that the product relation is redundant here: any solution $(a_1, \ldots, a_n)$ to (1) will satisfy $a_1 \ldots a_n = 1$. To see this, let $y \coloneqq a_1 \ldots a_n$. Then the recursive relations in (1) yield $y = (y^{a_n^{-1}}, 1, \ldots, 1)$, i.e., $y$ is a fixed point of the contracting function $F(w) \coloneqq (w^{a_n^{-1}}, 1, \ldots, 1)$, which already has the identity as a fixed point. By Proposition **??**, we must then have $y = 1$. They exist.

**Theorem 3.2** (Self-similarity)**.** *Model groups* $M$ *are self-similar, strongly fractal, and closed under taking diagonals. That is,*

*(i)* $\operatorname{Stab}_1 M \leq M^d$.

*(ii)* $\Delta M \leq M$.

*(iii) For each* $1 \leq j \leq d$, $\pi_j(\operatorname{Stab}_1 M) = M$.

*Proof.* The first assertion follows immediately from the recursive definitions of the generators of $M$. Specifically, the recurrences imply that $a_i \in M^d \rtimes \langle\sigma\rangle$ for each $1 \leq i \leq n$, and hence $M \leq M^d \rtimes \langle\sigma\rangle$. Thus,

$$\operatorname{Stab}_1 M \leq \operatorname{Stab}_1(M^d \rtimes \langle\sigma\rangle) = M^d.$$

For (ii), note first that $\Delta M$ is generated by the elements $(a_i, \ldots, a_i)$, $1 \le i \le n$. Using the recursive relations (1) once more, we find that

$$(a_n, \ldots, a_n) = a_1^d \in M,$$

and

$$(a_m, \ldots, a_m) = a_{m+1}^d \in M.$$

Now, for $1 < i \le m$,

$$(a_{i-1}, \ldots, a_{i-1}) = a_i a_i^{a_1} \ldots a_i^{a_1^d} \in M,$$

and for $m+1 < i \le n$,

$$(a_{i-1}, \ldots, a_{i-1}) = a_i a_i^{a_{m+1}} \ldots a_i^{a_{m+1}^d} \in M.$$

Thus, $(a_i, \ldots, a_i) \in M$ for all $1 \le i \le n$. This proves the second statement.

The final assertion now follows from the (i) and (ii): since $\Delta M \le \operatorname{Stab}_1 M \le M^d$, we have

$$M = \pi_j(\Delta M) \le \pi_j(\operatorname{Stab}_1 M) \le \pi_j(M^d) = M.$$

□

### 3.2 Semirigidity of model groups

**Proposition 3.3** (Invariant generation)**.** *Suppose $G = \langle\!\langle a_1^{g_1}, \ldots, a_n^{g_n} \rangle\!\rangle$, for arbitrary $g_1, \ldots, g_n \in M$. Then $G = M$.*

*Proof.* Since both $G$ and $M$ are closed (and hence profinite) subgroups of $W$, it suffices to show that $G_r = M_r$ for all $r \ge 0$. To achieve this, we inductively prove the following assertion for $r \ge 0$:

$(*_r)$: For all $g_1, \ldots, g_n \in M$, we have $\langle\!\langle a_1^{g_1}, \ldots, a_n^{g_n} \rangle\!\rangle =_r M$.

The base case $r = 0$ is trivial, since both groups in question are trivial on the zeroth level. Now assume $(*_r)$ holds for some $r \ge 0$. Take arbitrary $g_1, \ldots, g_n \in M$, and let $G \coloneqq \langle\!\langle a_1^{g_1}, \ldots, a_n^{g_n} \rangle\!\rangle$. To prove that $G =_{r+1} M$, it suffices to show that $M \le_{r+1} G$, as the reverse inclusion follows from $G \le M$.

By conjugating everything with $g_1^{-1}$, we may assume, without loss of generality, that $g_1 = 1$, i.e., $a_1 \in G$. Then conjugating the generators of $G$ with (possibly distinct) powers of $a_1$ does not change $G$. Hence, we may also assume, without loss of generality, that $g_2, \ldots, g_n \in \operatorname{Stab}_1 M$.

Now, we show that $a_i \in_{r+1} G$ for all $1 \le i \le n$, which by the above suffices to conclude the induction. We already have $a_1 \in G$. Let $w_i \coloneqq g_i|_d$ for $1 < i \le m$, and $w_i \coloneqq g_i|_1$ for $m+1 < i \le n$, and note that $w_i \in M$ by Proposition 3.2. Note also that for $1 < i \le m$, we have

$$a_i^{g_i} = (1, \ldots, 1, a_{i-1}^{w_i}),$$

and for $m+1 < i \le n$, we have

$$a_i^{g_i a_1} = (1, \ldots, 1, a_{i-1}^{w_i a_n}).$$

This means that $\Gamma \coloneqq \pi_d(\operatorname{Stab}_1 G)$ contains an $M$-conjugate of $a_i$, $i \in \{1, \ldots, n\} \backslash \{m, n\}$. Since $a_1 \in G$, we have $(a_n, \ldots, a_n) = a_1^d \in G$, and hence $a_n \in \Gamma$. Similarly, we have

$$\left(a_{m+1}^{g_{m+1}}\right)^d = \left(a_{m+1}^d\right)^{g_{m+1}} = \left(a_m^{g_{m+1}|_1}, \ldots, a_m^{g_{m+1}|_d}\right),$$

meaning $a_m^{g_{m+1}|_d} \in \Gamma$. The inductive hypothesis $(*_r)$ then implies that $\Gamma =_r M$. Now, for $1 < i \leq m$, there exists a $\gamma_i \in \mathrm{Stab}_1 G$ such that $\gamma_i|_d =_r w_i^{-1}$. Hence,

$$G \ni a_i^{g_i \gamma_i} = \left(1, \dots, 1, a_{i-1}^{w_i \gamma_i|_d}\right) =_{r+1} (1, \dots, 1, a_{i-1}) = a_i.$$

Similarly, for $m+1 < i \leq n$, there exists a $\gamma_i \in \mathrm{Stab}_1 G$ such that $\gamma_i|_d =_r (w_i a_n)^{-1}$. Hence,

$$G \ni a_i^{g_i a_1 \gamma_i} = \left(1, \dots, 1, a_{i-1}^{w_i a_n \gamma_i|_d}\right) =_{r+1} (1, \dots, 1, a_{i-1}) = a_i^{a_1^{1-d}},$$

which means $a_i \in_{r+1} G$. We have thus shown that for $i \in \{1, \dots, n\} \backslash \{m+1\}$, we have $a_i \in_{r+1} G$. From the product relation $a_1 \dots a_n = 1$, we conclude that $a_{m+1} \in_{r+1} G$. This proves $M \leq_{r+1} G$, concluding both the induction and the proof of the proposition. □

**Theorem 3.4** (Semirigidity)**.** *Suppose $b_1, \dots, b_n$ are $W$-conjugates of $a_1, \dots, a_n$, respectively. Then there exist $w \in W$ and $u_1, \dots, u_n \in M$ such that for $1 \leq i \leq n$,*

$$b_i^w = a_i^{u_i}.$$

*In particular,*

$$\langle\langle b_1, \dots, b_n \rangle\rangle^w = M.$$

*Proof.* Fill this

Induct on the claim $(*_\ell)$: there exists some $w_\ell \in W$ and assorted $u_{i,\ell} \in A$ such that

$$b_i^{w_\ell} = a_i^{u_{i,\ell}}.$$

Note that the sets of such $w$'s and $u$'s for each $\ell$ are nested and closed/compact/whatever, so as soon as we prove that each is nonempty then we can find something – in other words, a limit gives a $w$ and $u_i \in A$ which do the above on all levels.

In reality this means there's a nice contracting function carving out the $w$s and $u$s together but it seems easier (for now) to write it out as an induction.

Base case: $\ell = 0$ is trivial, or $\ell = 1$ if you are feeling cautious.

So assume we have elements as above for $\ell$ and we want to lift them up to level $\ell + 1$. Set $w_{\ell+1} = (w_\ell, ..., w_\ell)$. For the non-special generators, we have

$$\begin{aligned} b_i^{w_{\ell+1}} &= (..., b_{i-1}^{x_{i-1} w_\ell}, ...), \\ &= (..., b_{i-1}^{w_\ell (x_{i-1}^{w_\ell})}, ...), \\ &=_{\ell+1} (..., a_{i-1}^{u_{i-1,\ell}(x_{i-1}^{w_\ell})}, ...). \end{aligned}$$

By induction, $x_{i-1}^{w_\ell} \in_\ell A$ because $x_{i-1} \in B$. Conjugating by an appropriate power of $a_1$ (depending on $i$ only, not $\ell$, although that seems irrelevant), we may assume that the nontrivial coordinate is correctly located, although this may modify the inner element by some further conjugation,

$$b_i^{w_{\ell+1} a_1^{k_i}} = (..., a_{i-1}^{u_{i-1,\ell}(x_{i-1}^{w_\ell}) v_i}, ...).$$

Since $A$ is fractal, it has, at level $\ell + 1$, an element of the form

$$u'_{i,\ell+1} :=_{\ell+1} (..., u_{i-1,\ell}(x_{i-1}^{w_\ell}) v_i, ...)^{-1}.$$

Comparing to the above, we see that

$$b_i^{w_{\ell+1} a_1^{k_i} u'_{i,\ell+1}} =_{\ell+1} (..., a_{i-1}, ...) = a_i.$$

So we can use $u_{i,\ell+1} := (a_1^{k_i} u'_{i,\ell+1})^{-1}$.

A similar calculation works for $b_1$, where the coordinate does not even need to be moved , though now $u_{1,\ell+1}$ will be a diagonal element instead; that's ok by prop/lemma X on diagonal closure.

Note that we've now shown all $b_i^w$ are in$_{\ell+1}$ $A$, so the product formula at least requires $b_{m+1}^w \in_{\ell+1} A$. But we still need to do a little more work to show that it's actually conjugate to $a_{m+1}$.

Luckily (tbd) painful calculation from the product formula shows that there is a $c \in B$ such that

$$b_{m+1}^c = (--, \square, 1)\sigma^{-1}$$

But by number theory, each coordinate of $(b_{m+1}^c)^d = (\square, ..., \square)$ is conjugate (in $B$) to $b_m$, so actually

$$b_{m+1}^c = (--, b_m^{x_m}, 1)\sigma^{-1}.$$

So now we can repeat versions of the games above and show that there's a $v_{m+1,\ell+1} \in A$ such that

$$(b_{m+1}^c)^{w_{\ell+1}} =_{\ell+1} a_{m+1}^{v_{m+1,\ell+1}}.$$

How to handle $c$? Well, we already proved that $w_{\ell+1}$ conjugates$_{\ell+1}$ $B$ into $A$, meaning $c^{w_{\ell+1}} \in_{\ell+1} A$, and moreover

$$b_{m+1}^{cw_{\ell+1}} =_{\ell+1} b_{m+1}^{w_{\ell+1} c^{w_{\ell+1}}}$$

Rearrange and let $u_{m+1,\ell+1} = v_{m+1,\ell+1} \tilde{c}^{-w_{\ell+1}}$ to get what we wanted, where $\tilde{c}^{w_{\ell+1}}$ is anything in $A$ that agrees$_{\ell+1}$ with $c^{w_{\ell+1}}$. □

# 4 Abelianization

# 5 Outer Action