Fourier-Wigner Transform and Signal Recovery

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Abstract

Signal recovery is a central problem in harmonic analysis, concerned with determining when a signal can be uniquely reconstructed from incomplete information. Much of the existing literature approaches this problem through the Fourier transform, deriving recovery conditions based on the support of a signal and that of its Fourier transform. In this thesis, we take a different perspective inspired by ideas from quantum mechanics. We replace the Fourier transform with the discrete analogue of the Wigner–Weyl transform, known as the Fourier–Wigner transform. In physics, the Wigner transform provides a phase-space representation of a function, encoding information in both position and momentum variables simultaneously. In the discrete setting, the Fourier–Wigner transform similarly represents a signal in both its spatial and Fourier domains at once. Our goal is to investigate how this phase-space viewpoint can be applied to problems in signal recovery. In particular, we study structural and uncertainty properties of the Fourier–Wigner transform and apply tools such as Bourgain's theorem to derive recovery conditions.

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1 Introduction

1.1 Basics of Signal Recovery

A signal is a function mapping $\mathbb{Z}_N$ to $\mathbb{C}$, which encodes a message and is transmitted to some receiver. This is typically done by mapping $\mathbb{Z}_N$ to $\{0, 1\}$, which are known as bits, and the full image of the signal is required to decipher the message. However, the process of transmission is not free of losses, which means parts of the image of the signal are lost in transmission. As a result of this, the message is undecipherable and is therefore lost. Since such losses are not-uncommon in our processes of transmissions, it is important to find conditions under which we can completely recover the transmitted signal.

The problem is actually impossible at face-value. Given a signal $f : \mathbb{Z}_N \to \{0, 1\}$, suppose $S \subset \mathbb{Z}_N$ is a set of values which is lost, with size $|S| = n$. Then it follows that there are 2^n possibilities for what the full signal f could be, which makes it virtually impossible to recover the signal. However, Donoho and Stark [5] were able to show that under some knowledge of the incomplete signal and the Fourier transform, it may be possible to recover the signal uniquely. We can state this more precisely below

Theorem 1. *(Donoho-Stark uncertainty principle [5, 9]) Suppose that $f : \mathbb{Z}_N \to \mathbb{C}$ is a signal supported on E and the Fourier transform ($\hat{f}$) of the signal is supported on S. Then*

$$|E| \cdot |S| \geq N \tag{1}$$

Remark. This theorem tell us that every signal has a minimum structure, in that it must satisfy the uncertainty principle. This was the key realization for signal recovery, the fact that every signal must have this inherent structure means we already have non-trivial hints through which we can determine the original signal.

The following example illustrates how the uncertainty principle applies to recovery [11].

Example 1.1. Suppose f is a signal supported on E and we transmit the Fourier transform of this signal, but some values are lost. Let S be a set such that

$$\{f(x)\}_{x \in S} \text{ is missing}$$

If f cannot be recovered uniquely, there must be a competing signal g supported on a set F with $|F| = |E|$. This signal must match f, for every m not in S

$$\hat{f}(m)|_{m \notin S} = \hat{g}(m)|_{m \notin S}$$

Define $h = f - g$, then it follows that it must be supported on a set of maximum size $2|E|$.

Furthermore, by linearity of the Fourier transform $\hat{h} = \hat{f} + \hat{g}$ which implies that $\hat{h}$ is only non-zero on S. By the uncertainty principle

$$|E| \cdot |S| \geq \frac{N}{2}$$

this means that if $|S| < \frac{N}{2|E|}$ it follows immediately that h is identically 0 and f can be recovered uniquely.

Since the uncertainty principle for signals was first published, the field of signal recovery has grown exponentially. Far stronger uncertainty principles have been developed by means of other inequalities and assumptions on the signal [9], and other recovery methods and algorithms have been developed as well [4]. Much of the theory of signal recovery relies entirely on the Fourier transform and it's properties, which are extensively covered in [9]. In this thesis, we instead propose using the Fourier-Wigner transform as a tool for signal recovery.

1.2 Physics Interlude

The principles of signal recovery ties in beautifully into quantum mechanics, and studying some approaches in quantum mechanics can prove very useful to signal recovery. In fact the basis of this thesis is to apply an approach that was originally developed in the context of quantum mechanics to the problem of signal recovery.

Quantum mechanics is a study of objects which are sufficiently small [7]. For objects that are described by quantum mechanics (particles), measurements for fundamental properties such as position or momentum are probabilistic rather than deterministic [10]. That is, the position of a particle is not necessarily localized, which we would expect for larger objects. These probability distributions for the particles are known as wave functions. Position and momentum are known as operators, which transform the wave functions that give the probability distribution associated with these variables. A foundational underlying principle of quantum mechanics is that the operators for positions and momentum do not commute. That is, they cannot be known with full certainty simultaneously. This is succinctly described by the Heisenberg uncertainty principle [8]

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Where Δx (Δp) is the uncertainty in position (momentum). Notice the similarity this has to the Donoho-Stark uncertainty principle. This connection is solidifies upon realizing that the transformation of the wave function from position space to momentum space is precisely the Fourier transform [10]

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx\, e^{-\frac{ipx}{\hbar}} \phi(x)$$

Where $\phi(x)$ ($\phi(p)$) is position (momentum) wave function. This enables and justifies us to think about signal recovery through some of the machinery of quantum mechanics.

In an attempt to consolidate quantum and classical mechanics, the Wigner-Weyl transformation was developed which transform a position wave function into a quasi-probability distribution depending on both position and momentum simultaneously [4]. The discrete analog of this is known as the Fourier-Wigner transformation which we will explore in this thesis. We will systematically develop some of the core properties first and then translate them into signal recovery.

2 Preliminaries

2.1 Definition and Notation

Let $f : \mathbb{Z}_N \to \mathbb{C}$ be a signal. We will define $\|\cdot\|_q$ to be the norm, with respect to the counting measure ($\mu(\{x\}) = 1$, where x is a singleton). That is

$$\|f\|_q = \left(\sum_{x=0}^{N-1} |f(x)|^q \right)^{\frac{1}{q}}$$

We will also use the normalized norm $\|f\|_{L^q(\mu)}$ which is defined as

$$\|f\|_{L^q(\mu)} = \left(\frac{1}{N^d} \sum_{x=0}^{N-1} |f(x)|^q \right)^{\frac{1}{q}}$$

Where d is the dimension of the domain of the function. It is worth noting that if f represents the the signal, then $d = 1$, and if f represents the autocorrelation, $d = 2$.

Definition 2.1. (Auto-correlation function): Given a signal $f : \mathbb{Z}_N \to \mathbb{C}$, define the auto-correlation of the signal $F : \mathbb{Z}_N^2 \to \mathbb{C}$ as

$$F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = f\left(x + \frac{y}{2}\right) \overline{f\left(x - \frac{y}{2}\right)} \tag{2}$$

Remark. One important thing to notice is that in the above definition is it samples values at a distance of $\pm y/2$ away from the midpoint x. If we define $u = x + \frac{y}{2}$ and $v = x - \frac{y}{2}$, this is a bijection with $u - v = y$ and $\frac{u+v}{2} = x$. This makes it obvious that the autocorrelation function $f(u, v)$ fixes a center point and compares how the function behaves at fixed distances away form this point. The importance of this will be made clear after the next definition.

Definition 2.2. (Fourier-Wigner transform): Given a signal $f : \mathbb{Z}_N \to \mathbb{C}$, we define the Fourier-Wigner Transform (FW) as

$$W_F(x, m) = N^{\frac{-1}{2}} \sum_{y \in \mathbb{Z}_N} F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \chi(-y \cdot m) \tag{3}$$

Where $\chi(-y \cdot m)$ is the character function $\chi(-y \cdot m) = e^{-2\pi i (y \cdot m)/N}$.

Remark. It is worth stating (in the context of quantum mechanics) that the Heisenberg uncertainty principle requires that one cannot know both the exact position and momentum of a particle simultaneously. However, W_F seems to do exactly that, by giving us both representations in the same function. This is precisely why we had to use the auto-correlation function instead of the signal it self. By sampling about a mid point, we "spread the support" in position space out, which means we create an uncertainty in the position which in turn allows us to then represent the momentum space function with more precision.

The following examples illustrates how the Fourier-Wigner transform treats familiar signals differently than the Fourier transform.

Example 2.1. Consider the signal $f(x) = e^{ik_1x} + e^{ik_2x} = \chi(k_1x) + \chi(k_2x)$, with $k_1 = \frac{2\pi n}{N}$ and $k_2 = \frac{2\pi m}{N}$. Clearly, the Fourier transform of the function is $\hat{f}(m) = \delta(k_1 - m) + \delta(k_2 - m)$, where δ is the Kronecker Delta function. However, if we compute the autocorrelation, we find that

$$\begin{aligned} F\left(x+\frac{y}{2}, x-\frac{y}{2}\right) &= \left(e^{ik_1(x+\frac{y}{2})} + e^{ik_2(x+\frac{y}{2})}\right)\left(e^{-ik_1(x+\frac{y}{2})} + e^{-ik_2(x+\frac{y}{2})}\right) \\ &= e^{ik_1y} + e^{ik_2y} + (e^{i(k_2-k_1)x} + e^{-i(k_2-k_1)x})e^{i\frac{(k_2+k_1)}{2}y} \end{aligned}$$

Then taking the FW transform, we find that

$$\begin{aligned} W_F(x,m) &= N^{\frac{-1}{2}} \sum_{y\in\mathbb{Z}_N} \left[e^{ik_1y} + e^{ik_2y} + (e^{i(k_2-k_1)x} + e^{-i(k_2-k_1)x})e^{i\frac{(k_2+k_1)}{2}y}\right]\chi(-y\cdot m) \\ &= N^{\frac{-1}{2}} \sum_{y\in\mathbb{Z}_N} \left[\chi([k_1-m]\cdot y) + \chi([k_2-m]\cdot y) + \cos\left(\frac{k_1+k_2}{2}x\right)\chi\left(y\left[\frac{k_1+k_2}{2} - m\right]\right)\right] \end{aligned}$$

We can solve this by using the following trick. Suppose that $a \neq 0$, then clearly $\chi(a) \neq 1$. We can use a telescoping sum as follows

$$\begin{aligned} (1-\chi(a))\sum_{y\in\mathbb{Z}_N}\chi(ay) &= \chi(a)\sum_{y\in\mathbb{Z}_n}\chi(ay) - \sum_{y\in\mathbb{Z}_n}\chi(ay) \\ &= \sum_{y\in\mathbb{Z}_n}\chi((a+1)y) - \sum_{y\in\mathbb{Z}_n}\chi((ay)) \\ &= \chi(Ny) - \chi(0) \\ &= 0 \end{aligned}$$

Which is only possible if $\sum \chi(ay) = 0$. On the other hand if $a = 0$, then $\chi(a) = 1$ which means that $\sum_{y\in\mathbb{Z}_N}\chi(a) = N$. Hence

$$W_F(x,m) = \delta(k_1-m) + \delta(k_2-m) + \cos\left(\frac{k_1+k_2}{2}x\right)\cdot\delta\left(\frac{k_1+k_2}{2} - m\right)$$

The last term appear to be an interference effect between the two signals, which does not appear with the regular Fourier transform. This has an important effect in quantum mechanics, since this term encodes the interference effect between quantum states, since we can create a superposition over two distinct states.

Example 2.2. Another interesting example to consider is a chirp signal defined as $f(x) = \chi(a \cdot x^2) = \exp(2\pi i a x^2/N)$, with $a \in \mathbb{Z}_N$. Then the autocorrelation is given by

$$F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = e^{\frac{2\pi i}{N} a(2xy)}$$

Which gives us that

$$W_F(x, m) = N^{\frac{1}{2}} \delta(2ax - m)$$

So the Fourier-Wigner transform behaves as a straight line in the support.

2.2 Fourier Transform Symmetries

We can explore some basic properties of the Fourier-Wigner transform which are symmetrical to properties of the Fourier transform. We expect at least a few properties to be prevalent in both since they share the same character function. We begin with a formula for inverting the transform.

Lemma 2. *(Inversion) The inversion formula for the FW transform is given by*

$$F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = N^{\frac{-1}{2}} \sum_{m \in \mathbb{Z}_n} W_F(x, m) \chi(y \cdot m) \tag{4}$$

Proof. This formula follows via a simple computation

$$\begin{aligned} W_F(x, m) &= N^{\frac{-1}{2}} \sum_{y \in \mathbb{Z}_N} F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \chi(-y \cdot m) \\ &= N^{-1} \sum_{y \in \mathbb{Z}_N} \sum_{m' \in \mathbb{Z}_N} W_F(x, m) \chi(y \cdot (m - m')) \\ &= \sum_{m' \in \mathbb{Z}_N} W_F(x, m) \delta(m - m') = W_F(x, m) \end{aligned}$$

Where the third line follows since $\sum_y \chi(y \cdot m) = N$ if $m = 0$ and the sum is 0 otherwise □

Remark. The structure of the Fourier-Wigner transform and it's inverse is quite similar to the Fourier transform. This suggests that some basic properties such as the Plancherel identity may also be true of the Fourier-Wigner transform.

Lemma 3. *(Projection Property) Given a signal f and it's FW transform W_F, we have that*

$$N^{\frac{-1}{2}} \sum_{m \in \mathbb{Z}_N} W_F(x,m) = |f(x)|^2 \qquad N^{\frac{-1}{2}} \sum_{x \in \mathbb{Z}_N} W_F(x,m) = |\hat{f}(m)|^2 \tag{5}$$

Proof. It follows from direct computation that

$$\begin{aligned} N^{\frac{-1}{2}} \sum_{m \in \mathbb{Z}_N} W_F(x,m) &= N^{-1} \sum_{m \in \mathbb{Z}_N} \sum_{y \in \mathbb{Z}_N} f\left(x + \frac{y}{2}\right) \overline{f\left(x - \frac{y}{2}\right)} \cdot \chi(-y \cdot m) \\ &= \sum_{y \in \mathbb{Z}_N} f\left(x + \frac{y}{2}\right) \overline{f\left(x - \frac{y}{2}\right)} \cdot \delta(y) \\ &= |f(x)|^2 \end{aligned}$$

As for the other one

$$\begin{aligned} N^{\frac{-1}{2}} \sum_{x} W_F(x,m) &= N^{-1} \sum_{x} \sum_{y} f\left(x + \frac{y}{2}\right) \overline{f\left(x - \frac{y}{2}\right)} \cdot \chi(-y \cdot m) \\ &= N^{-1} \sum_{u} \sum_{v} f(u)\overline{f(v)} \cdot \chi((v-u) \cdot m) \end{aligned}$$

Where we have introduced the change of coordinates $u = x + \frac{y}{2}, v = x - \frac{y}{2}$. Since we are summing over all x and y values, we can switch the indices of the sum from $\sum_{x,y}$ to $\sum_{u,v}$. Then

$$\begin{aligned} N^{-1} \sum_{u} \sum_{v} f(u)\overline{f(v)} \cdot \chi((v-u) \cdot m) &= N^{-1} \left(\sum_{u} f(u)\chi(-u \cdot m) \right) \left(\sum_{v} f(v)\chi(-v \cdot m) \right) \\ &= N^{-1} \left| \sum_{u} f(u)\chi(-u \cdot m) \right|^2 \\ &= |\hat{f}(m)|^2 \end{aligned}$$

□

Remark. Notice that this property of the FW transform immediately implies the Plancherel identity for the Fourier transform. We can continue our our discussion of how this system mirrors the Fourier system by proving Plancherel identity for the FW transform.

Lemma 4. *(Plancherel Identity) : Let be F be an autocorrelation function. Then*

$$\|W_F\|_2 = \|F\|_2 \tag{6}$$

Proof. For the purpose of space, let $F(x + \frac{y}{2}, x - \frac{y}{2})$ be denoted by $F(x,y)$. Then by the inversion formula, we can write $|W_F|^2$ as the product of two sums with different indices

$$\begin{aligned}\|W_F\|_2 &= \left(N^{-1}\sum_{x,m}\sum_{y,y'} F(x,y)\overline{F(x,y')}\cdot\chi(-(y-y')m)\right)^{\frac{1}{2}} \\ &= \left(\sum_{x}\sum_{y,y'} F(x,y)\overline{F(x,y')}\cdot\delta(y'-y)\right)^{\frac{1}{2}} \\ &= \left(\sum_{x}\sum_{y} |F(x,y)|^2\right)^{\frac{1}{2}} = \|F\|_2\end{aligned}$$

□

Remark. In the process of proving this we actually derived a different identity, by noticing that the sum over x did not participate at all. That is, in the process of the proof above, we actually showed that

$$\sum_{m\in\mathbb{Z}_N} |W_F(x,m)|^2 = \sum_{y\in\mathbb{Z}_N}\left|F\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\right|^2 \tag{7}$$

This will ultimately be key in helping us apply the Fourier-Wigner transform, by letting us derive an uncertainty principle for this system, similar to the Donoho-Stark uncertainty principle [5].

Theorem 5. *(Uncertainty principle for FW) Let f be a signal. Suppose that $W_F(x,m)$ is only supported when $m \in A$ and $F(x+y/2, x-y/2)$ is only supported when $y \in B$. Then*

$$N \le |A|\cdot|B| \tag{8}$$

Proof. By the inversion formula

$$\left|F\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\right|^2 = \left|N^{-\frac{1}{2}}\sum_{m\in\mathbb{Z}_N} W_F(x,m)\,\chi(y\cdot m)\right|^2$$

Since $W_F(x,m)$ is non-zero only when $m \in A$, we can sum over only these values instead. Then, by Cauchy-Schwarz

$$\begin{aligned}\left|F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\right|^2 &= \left|N^{-\frac{1}{2}}\sum_{m\in A} W_F(x,m)\ \chi(y\cdot m)\right|^2\\ &\leq N^{-1}\left(\sum_{m\in A}|W_F(x,m)|^2\right)\cdot|A|\\ &\leq N^{-1}\left(\sum_{m\in\mathbb{Z}_N}|W_F(x,m)|^2\right)\cdot|A|\\ &= N^{-1}\left(\sum_{y\in\mathbb{Z}_N}\left|F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\right|^2\right)\cdot|A|\end{aligned}$$

In the third line, we have added back in the values $m\notin A$, since they contribute 0 to the overall sum. This also allows us to use the inversion formula. The last line follows by the previous remark. The sum in the last line is only non-zero when $y\in B$. Hence we can remove all $y\notin B$ from the sum. Finally, if we sum over $y\in B$ on both sides it follows that

$$\sum_{y\in B}\left|F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\right|^2 \leq N^{-1}\left(\sum_{y\in B}\left|F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\right|^2\right)\cdot|A||B|$$

$$N\leq|A|\cdot|B|$$

□

Remark. This uncertainty principle is different from the Donoho-Stark principle, as it relates the support in the two-point correlation variable y to the support on the phase space variable m, for a fixed slice of x. This is actually an uncertainty principle on the phase space itself, rather than just the supports of the two functions. The motivation of this theorem is that it gives us a fundamental condition on the support of the auto-correlation function and the Fourier-Wigner transform.

Theorem 6. *(Recovery [11]) Suppose that f is a signal with auto-correlation $F(x+y/2,x-y/2)$, supported when $y\in B$. Let $W_G(x,m):\mathbb{Z}_N^2\to\mathbb{C}$ be a function such that*

$$W_G(x,m)=\begin{cases}W_F(x,m) & m\notin A\\ 0 & m\in A\end{cases}$$

For $A\subset\mathbb{Z}_N$. Then provided that

$$|A|\cdot|B|<\frac{N}{2}$$

$F(x+y/2,x-y/2)$ (and by extension f) can be recovered exactly from W_G

Remark. The reason as to why this argument works is identical to the set up given in example 1.1. The motivation is simply that if we cannot recover the signal exactly, then there must be a second signal which matches our original signal everywhere except on the set where values are lost. If we take the difference of the two signals, we can construct a third signal which by the uncertainty principle must satisfy $|A| \cdot |B| \geq N/2$. However, if it is the case that $|A| \cdot |B| < N/2$, then it immediately follows that the third signal is identically 0, meaning that f and g must match. Exact recovery mechanisms and algorithm based on these kinds of uncertainty principles are also possible and given in [5, 9].

This concludes our initial discussion of symmetrical properties between the Fourier transform and the Fourier-Wigner transform and how this makes it a plausible candidate for signal recovery. We now deviate temporarily from the program to state an incredibly important result for harmonic analysis and signal recovery. This theorem will give us new machinery to develop more sophisticated results in signal recovery immediately.

3 Bourgain's Theorem and Applications

Here we will state a very well known and powerful result, known as Bourgain's theorem. The proof of this theorem will not be given here, but can be found in [3].

Theorem 7. *Bourgain's Theorem: Suppose G is locally compact and Abelian. Let $f_1, \ldots, f_n$ be a sequence of orthogonal functions with $\|f_i\|_{L^\infty(G)} \leq 1$. Then there exists a subset S with $|S| \geq n^{2/q}$ such that*

$$\Big\| \sum_{i \in S} a_i f_i \Big\|_{L^q(G)} \leq C(q) \Big(\sum_{i \in S} |a_i|^2 \Big)^{\frac{1}{2}}$$

Where the constant $C(q)$ depends only on q

To leverage this result, we need to find an orthogonal system for the FW transform. It is easy to check that the following system works

Lemma 8. *(Orthogonal set) The functions*

$$\psi_{k,x'} = N^{-\frac{1}{2}} \delta(x - x') \chi(k \cdot m)$$

with $x', k \in \{0, \ldots, N-1\}$ are an orthogonal system for the FW transform

Proof. The following computation is enough to show that $\psi_{k,x'}$ are mutually orthogonal

$$\begin{aligned}\langle \psi_{k,x'}, \psi_{k',x''} \rangle &= N^{-1} \sum_{x,m} \delta(x - x')\delta(x - x'')\chi((k' - k) \cdot m) \\ &= \delta_{x',x''} N^{-1} \sum_{m} \chi((k' - k) \cdot m) \\ &= \delta_{x',x''} \delta_{k,k'}\end{aligned}$$

□

This unlocks a corollary [9] from Bourgain's theorem which is stated below

Theorem 9. *Let f be a signal for a generic subset Σ of $\mathbb{Z}_N^2$ of size $N^{4/q}$, where $q > 2$. If $\hat{f}$ is supported in Σ, then*

$$\|f\|_{L^q(\mu)} \leq C(q)\|f\|_{L^2(\mu)} \tag{9}$$

Remark. For our purposes, a generic set is given by the condition that if we chose a set of size n, then the probability of any element in $\mathbb{Z}_N$ is $p = n^{\frac{2}{q}-1}$ [9]. Then equation 9 will hold for this set with a probability close to 1.

We can translate this into the following recovery condition

Theorem 10. *(Uncertainty principle using Bourgain's) Suppose that $f : \mathbb{Z}_N^2 \to \mathbb{C}$ is a signal satisfying 9 and let A be the support of f. Then*

$$\frac{N^2}{C(q)^{\frac{1}{\frac{1}{2}-\frac{1}{q}}}} \leq |A| \tag{10}$$

Proof. From 9 we know that

$$\|f\|_{L^q(\mu)} \leq C(q)\|f\|_{L^2(\mu)}$$

We can apply the following variation of Holder's inequality, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$

$$\|f\|_r \leq \|f\|_q \cdot \|1\|_p \tag{11}$$

Setting $r = 2$ and $\frac{1}{p} = \frac{1}{2} - \frac{1}{q}$, it follows that

$$\|f\|_{L^2(\mu_A)} \leq \|f\|_{L^q(\mu_A)} \left(\frac{|A|}{N^2}\right)^{\frac{1}{2}-\frac{1}{q}}$$

Since f is supported in A. If we combine this with the first inequality it follows that

$$\|f\|_{L^q(\mu)} \leq C(q)\|f\|_{L^q(\mu_A)} \left(\frac{|A|}{N^2}\right)^{\frac{1}{2}-\frac{1}{q}}$$

Canceling and re-arranging gives us the desired result □

The uncertainty principle above gives us a very strong recovery condition, however it still remains to show that there exists signals whose FW transform/autocorrelation satisfies this bound. We will now work with the case where $q = 4$ and try to see if we can construct conditions which will allow our function to satisfy theorem 9. As we will find out, the L^4 norm of the Fourier-Wigner transform is very deeply rooted in the additive structure of the set.

3.1 L^2 and L^4 Comparisons

In the following examples, we will explore the setting where the function (either auto-correlation or Fourier-Wigner) is supported on a Sidon set [6] which has the following feature. If $a, b, c, d \in A$ and A is Sidon, then

$$a + b = c + d \iff a = c \text{ or } a = d$$

This means our set has minimal additive structure. Examples of such sets are geometric progressions, sets such at $\{2^n\}_{n\in\mathbb{Z}^+}$. These sets will allow us to expose the dependence of additive energy (structure) on the L^4 norm of the Fourier-Wigner system.

Example 3.1. (*Sidon set*) Consider the autocorrelation of a signal, defined by f with a FW transform of $W_F(x, m) = h(x)1_B(m)$, where $h : \mathbb{Z}_N \to \mathbb{C}$ is an arbitrary function. From Plancherel, it directly follows that

$$\|F\|_{L^2(\mu)} = \|W_F\|_{L^2(\mu)} = |B|^{\frac{1}{2}}|N|^{-1}\|h\|_2$$

On the other hand, we can compute the L^4 norm with

$$\begin{aligned}
\|F\|_{L^4(\mu)}^4 &= N^{-2}\sum_{x,y}\left(F\left(x-\frac{y}{2},x+\frac{y}{2}\right)\right)^4 \\
&= N^{-4}\sum_{x,y}\sum_{m,m',l,l'}|h(x)|^4 1_B(m)1_B(m')1_B(l)1_B(l')\cdot\chi(y\cdot(m-m'+l-l')) \\
&= N^{-3}\sum_{x}|h(x)|^4\sum_{m,m',l,l'}1_B(m)1_B(m')1_B(l)1_B(l')\cdot\delta(m-m'+l-l') \\
&= N^{-3}\sum_{x}|h(x)|^4\sum_{m+l=m'+l'}1_B(m)1_B(m')1_B(l)1_B(l') \\
&= N^{-3}\|h\|_4^4\cdot|E(B)|
\end{aligned}$$

Where $E(B)=\{a,b,c,d\in B\mid a+b=c+d\}$ is known as the additive energy of B [1]. Then by Holder's inequality it follows that

$$\|h\|_2\leq N^{\frac{1}{4}}\|h\|_4$$

Using this we can conclude that $\|F\|_{L^4(\mu)}\leq\|F\|_{L^2(\mu)}$ holds for this function if

$$\begin{gathered}
N^{-\frac{3}{4}}|E(B)|^{\frac{1}{4}}\|h\|_4\leq|B|^{\frac{1}{2}}N^{-1}\|h\|_2 \\
|E(B)|\leq|B|^2
\end{gathered}$$

In the case where B is a Sidon set the additive energy is equal to $|B|^2$ and hence the Bourgain recovery condition holds for any function of this form.

Example 3.2. We can generalize this to non-indicator functions. Consider the case where f is a signal supported on a Sidon set. Further, suppose that $F(x+y/2,x-y/2)=h(x)g(y)$ (an example of such a signal would be $f(x)=\chi(a\cdot x^2)$). Then

$$\begin{aligned}
\|W_F\|_{L^2(\mu)} &= N^{-1}\left(\sum_{x,m}|W_F|^2\right)^{\frac{1}{2}} \\
&= N^{-1}\left(\sum_{x,m}\sum_{y,y'}F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\overline{F\left(x+\frac{y'}{2},x-\frac{y'}{2}\right)}\cdot\chi((y'-y)\cdot m)\right)^{\frac{1}{2}} \\
&= N^{-\frac{1}{2}}\left(\sum_{x}\sum_{y}|h(x)g(y)|^2\right)^{\frac{1}{2}} \\
&= N^{\frac{-1}{2}}\|h\|_2\cdot\|g\|_2
\end{aligned}$$

We can perform the same computation with the L^4 norm

$$\begin{aligned}\|W_F\|_{L^4(\mu)} &= N^{-1}\left(\sum_{x,m}|W_F|^4\right)^{\frac{1}{4}} \\ &= N^{-1}\left(\sum_{x,m}\sum_{y,y',z,z'}|h(x)|^4\cdot|g(y)\,g(y')\,g(z)\,g(z')|\,\chi(-(y+y'-z-z')\cdot m)\right)^{\frac{1}{4}} \\ &= N^{-\frac{3}{4}}\left(\sum_{x}\sum_{y+y'=z+z'}|h(x)|^4\cdot|g(y)\,g(y')\,g(z)\,g(z')|\right)^{\frac{1}{4}}\end{aligned}$$

Unlike the previous example, we have to be a bit more careful in how we sum over the terms. Since our set has Sidon structure, the sum splits as

$$\begin{aligned}\sum_{y+y'=z+z'}|g(y)\,g(y')\,g(z)\,g(z')| &= \sum_{y,y'}|g(y)|^2|g(y')|^2+\sum_{y,z}|g(y)|^2|g(z)|^2-\sum_{y}|g(y)|^4 \\ &= 2\left(\sum_{y}|g(y)|^2\right)^2-\sum_{y}|g(y)|^4 \qquad (12)\end{aligned}$$

Where the last sum subtracts away the terms which are double counted, which is when $y=y'=z=z'$. Then it follows that

$$\begin{aligned}\|W_F\|_{L^4(\mu)} &\leq N^{\frac{-3}{4}}\left(2\sum_{x}|h(x)|^4\left(\sum_{y}|g(y)|^2\right)^2\right)^{\frac{1}{4}} \\ &= 2^{\frac{1}{4}}\cdot N^{\frac{-3}{4}}\|h\|_4\|g\|_2\end{aligned}$$

Then it becomes clear that for $\|W_F\|_{L^4(\mu)}\leq C\|W_F\|_{L^2(\mu)}$ to hold we need that

$$\|h\|_4\leq N^{\frac{1}{4}}\|h\|_2$$

Which is equivalent to the condition that $\|h\|_{L^4(\mu)}\leq\|h\|_{L^2(\mu)}$

Example 3.3. We can construct our most general possible example if we try to find a conditions to compare the L^4 and L^2 norms of the auto correlation f instead. As we will see, the clear separation of the x,m coordinates will allow for more smooth manipulations when we use the Sidon structure. As before by Plancherel, $\|f\|_{L^2(\mu)}=\|W_F\|_{L^2(\mu)}$. Suppose that W_F is supported on a set with Sidon structure. Then

$$
\begin{aligned}
\|F\|_{L^4(\mu)} &= \left(N^{-2}\sum_{x,y}\left|F\left(x+\frac{y}{2},x-\frac{y}{2}\right)\right|^4\right)^{\frac{1}{4}} \\
&= \left(N^{-4}\sum_{x,y}\sum_{m,m',l,l'} W_F(x,m)W_F(x,m')\overline{W_F(x,l)W_F(x,l')}\ \chi(-y\cdot(m+m'-l-l')\right)^{\frac{1}{4}} \\
&= \left(N^{-3}\sum_{x}\sum_{m+m'=l+l'} W_F(x,m)W_F(x,m')\overline{W_F(x,l)W_F(x,l')}\right)^{\frac{1}{4}} \\
&\leq 2^{\frac{1}{4}}\left(N^{-3}\sum_{x}\left(\sum_{m}|W_F(x,m)|^2\right)^2\right)^{\frac{1}{4}} \qquad (13)
\end{aligned}
$$

Where the last line follows by applying equation 12 to the inner sum. Applying Cauchy-Schwarz [2] to the sum over m it follows that

$$
\begin{aligned}
\left(N^{-3}\sum_{x}\left(\sum_{m}|W_F(x,m)|^2\right)^2\right)^{\frac{1}{4}} &\leq \left(N^{-3}\left(\sum_{x,m}|W_F(x,m)|^4\right)\cdot N\right)^{\frac{1}{4}} \\
&= \|W_F\|_{L^4(\mu)}
\end{aligned}
$$

Hence on a Sidon set $\|F\|_{L^4(\mu)} \leq C\|F\|_{L^2(\mu)}$ if $\|W_F\|_{L^4(\mu)} \leq C\|W_F\|_{L^2(\mu)}$

Remark. The previous example exposes the power of the Sidon structure, in that the Bourgain result for the auto-correlation function is guaranteed if it holds for the Fourier-Wigner transform of the function.

3.2 The Additive Energy Connection

The additive energy of a set is loosely a measure of the structure of a set, where as we will see, sets with strong energy are unstructured and weak energy are highly structured. As we showed previously, in the case of the L^4 norm, we always arrive at term involving

$$\sum_{a,b,c,d}(\dots)\ \delta(a+b-c-d)$$

which implies that we perform the sum over the additive energy of the set $E(B)$. This suggests to us that the additive energy may act as a controlling parameter for whether we are able to find functions satisfying $\|\cdot\|_{L^4(\mu)} \leq C\|\cdot\|_{L^2(\mu)}$, where the $\cdot$ represents either the FW transform or the autocorrelation of the function. In this section, we explore how the estimates change based on how the additive structure of the set changes.

Let F be an autocorrelation with the FW transform given by $W_F(x,m) = 1_A(x)1_B(m)$. Then by Plancherel $\|F\|_{L^2(\mu)} = \|W_F\|_{L^2(\mu)} = N^{-1}|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}$. For the L^4 norm, we can compute that

$$\begin{aligned}
\|F\|_{L^4(\mu)} &= \left(N^{-4} \sum_{x,m,m',l,l'} 1_A(x)1_B(m)1_B(m')1_B(l)1_B(l') \cdot \chi(y \cdot (m + m' - l - l')) \right)^{\frac{1}{4}} \\
&= \left(N^{-3} \sum_x 1_A(x) \sum_{m,m',l,l'} 1_B(m)1_B(m')1_B(m')1_B(l)1_B(l') \cdot \delta(m + m' - l - l') \right)^{\frac{1}{4}} \\
&= \left(N^{-3}|A| \cdot |E(B)| \right)^{\frac{1}{4}} \qquad (14)
\end{aligned}$$

We can now differently see how the different additive energies of sets affect the L^4 to L^2 comparisons.

Example 3.4. (*Sidon set*) Consider the case where B is a Sidon set, which means that $a+b = c+d$ satisfies $a = c$ or $a = d$. This means that selection of a and b uniquely determines c and d. This implies that

$$|E(B)| = C|B|^2$$

plus a constant. In this case, we can use 14 to find the condition for $\|F\|_{L^4(\mu)} \leq \|F\|_{L^2(\mu)}$ with this function is

$$\begin{gathered}
\left(N^{-3}|A| \cdot C|B|^2 \right)^{\frac{1}{4}} \leq N^{-1}|A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \\
CN \leq |A|
\end{gathered}$$

Remark. Notice that if $C = 1$ this means that $|A| = N$. By the uncertainty principle we derived, it follows that $|B| = 1$, which implies that the signal is localized to a single point. Notice also that in the argument the dependence on the size of the set B drops out entirely. This we also characteristic of 3.1, hence we can consider other functions with respect to x, and not worry about the set B.

Example 3.5. (*Random set*) We can also consider where B is a random set, which we expect to have no non-trivial additive structure. Define B to be random if

$$\mathbb{E}[x \in B] = p, \quad \forall x \in \mathbb{Z}_N$$

Consider the same function as before and let B be a random set. We can work out the L^4 norms with the same trick. Since B is random, we instead must compute in terms of the expected value for quantities involving B. Notice that the $\sum_x 1_A(x) = |A|$ and the sum over m, m', l, l' becomes the additive energy of B, $E(B)$. Proceeding with expected values, we are interested in

$$\begin{aligned}\mathbb{E}(|E(B)|) &= \mathbb{E}\left(\sum_{y+y'=z+z'} 1_B(y)1_B(y')1_B(z)1_B(z')\right)\\ &= \sum_{y+y'=z+z'} \mathbb{E}\left(1_B(y)1_B(y')1_B(z)1_B(z')\right)\\ &= \sum_{y+y'=z+z'} \mathbb{E}(1_B(y)) \cdot \mathbb{E}(1_B(y')) \cdot \mathbb{E}(1_B(z)) \cdot \mathbb{E}(1_B(z'))\end{aligned}$$

Where the second step follows by linearity of expectation value and the third step follows since each variable is independent. Finally by the definition of the indicator function and the random set it follows that

$$\sum_{y+y'=z+z'} \mathbb{E}(1_B(y)) \cdot \mathbb{E}(1_B(y')) \cdot \mathbb{E}(1_B(z)) \cdot \mathbb{E}(1_B(z')) = \sum_{y+y'=z+z'} p^4 = p^4|E(B)|$$

Now notice that a, b, c uniquely determines a d such that $a + b - c = d$. This implies that $|E(B)| \leq N^3$. We can finish by realizing that $|B| = pN$ which follows by the definition of a random set. Then

$$\mathbb{E}(|E(B)|) \leq \frac{|B|^4}{N}$$

Which is true up to constants or constant multiples. Then it follows that

$$\|W_F\|_{L^4(\mu)} \leq \left(N^{-3}|A| \cdot \mathbb{E}(|E(B)|)\right)^{\frac{1}{4}} = \left(N^{-4}|A| \cdot |B|^4\right)^{\frac{1}{4}}$$

Thus for $\|W_F\|_{L^4(\mu)} \leq \|W_F\|_{L^2(\mu)}$ we would need that

$$\begin{aligned}N^{-1}|A|^{\frac{1}{4}}|B| &\leq N^{-1}|A|^{\frac{1}{2}}|B|^{\frac{1}{2}}\\ |B|^2 &\leq |A|\end{aligned}$$

Remark. In this example we have a condition for the Bourgain estimate to go through requiring us to compare the sizes of the support for position and Fourier coordinates. Using 8, it follows that $|A| \geq N^{\frac{2}{3}}$ which means A can be a nontrivial set.

Example 3.6. *(Strong additive energy)* Consider the case where we have the strongest possible additive energy. In this case a, b, c uniquely determine a d such that $a+b-c = d$, then $E(B) = |B|^3$ up to a constant. Then

$$\|W_F\|_{L^4(\mu)} = \left(N^{-3}|A| \cdot |E(B)|\right)^{\frac{1}{4}} = \left(N^{-3}|A| \cdot |B|^3\right)^{\frac{1}{4}}$$

Comparing the two norms we have that $\|W_F\|_{L^4(\mu)} \leq \|W_F\|_{L^2(\mu)}$ if

$$N^{-3}|A||B|^3 \leq N^{-4}|A|^2|B|^2$$
$$N|B| \leq |A|$$

Remark. By the uncertainty bound, this again forces that $|A| = N$ and $|B| = 1$, hence the sets must be trivial again.

We can culminate these ideas into a final recovery condition using additive structure on one side, and the L^4 norm on the other, it the special case that F is an indicator function.

Theorem 11. *(Support Condition Using Additive Energy) Suppose $f : \mathbb{Z}_N \to \mathbb{C}$ is a signal, such that the autocorrelation is supported on A, with $F(x + y/2, x - y/2) = 1_A(x + y/2)1_A(x - y/2)$. Suppose that W_F is supported on a set with Sidon structure. Then*

$$\frac{N}{2} \leq |A| \tag{15}$$

Where c is a constant independent of the size of A.

Proof. From the previous examples, we have that

$$\begin{aligned}
\sum_{y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^4 &= N^{-1} \sum_{m+m=l+l'} W_F(x,m) W_F(x,m') \overline{W_F(x,l) W_F(x,l')} \\
&\leq 2N^{-1} \left(\sum_{m \in \mathbb{Z}_N} |W_F(x,m)|^2 \right)^2 \\
&= 2N^{-1} \left(\sum_{y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^2 \right)^2
\end{aligned}$$

Where the last step follows from the remarks after the Plancherel identity. Now notice that

$$\sum_{y \in \mathbb{Z}_N} F\left(x + \frac{y}{2}, x - \frac{y}{2}\right)^2 = \sum_{y \in \mathbb{Z}_N} 1_A\left(x + \frac{y}{2}\right) 1_A\left(x - \frac{y}{2}\right)$$

If we let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$, then we get the conditions that $u + v = 2x$ and $u - v = y$. Since this is a bijection, each y corresponds to a unique (u, v), which satisfies $u + v = 2x$. Then, if we sum over all y, it is equivalent to summing over all pairs with $u + v = 2x$. Then let

$$\begin{aligned} g(x) &= \sum_{y \in \mathbb{Z}_N} 1_A(x + \frac{y}{2}) 1_A(x - \frac{y}{2}) \\ &= \sum_{u,v \in \mathbb{Z}_N^2,\ u+v=2x} 1(u)1(v) \\ &= |\{u, v \in A \mid u + v = 2x\}| \end{aligned}$$

But then

$$\begin{aligned} \sum_{x \in \mathbb{Z}_N} g(x)^2 &= |\{u, v \in A \mid u + v = 2x\}| \cdot |\{a, b \in A \mid a + b = 2x\}| \\ &= |\{a, b, u, v \in A \mid a + b = u + v\}| = |E(A)| \end{aligned} \tag{16}$$

We can also sum over x on the RHS, which gives us that

$$\sum_{x,y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^4 = \sum_{u,v \in \mathbb{Z}_N} 1\,(u)1(v) = |A|^2$$

Putting both together, it follows that

$$|A|^2 = \|F\|_4^4 \leq 2N^{-1}|E(A)|$$

If we assume nothing special about A, then we can say that $E(A) \leq |A|^3$, from which it follows

$$\frac{N}{2} \leq |A|$$

□

Remark. This theorem proves a very strong result however it requires a correspondingly strong condition that $F(x + y/2, x - y/2) = 1_A(x + y/2)1_A(x - y/2)$. For the purposes of signal recovery, we often need to take the difference of two signals, which match everywhere but the set of lost values. It is easy to see that this does not often give us back an indicator function, hence making this theorem not usable directly for recovery. We would require weaker conditions of F, which is exactly what we do in the next argument, since much of the work was already done here.

Theorem 12. *(Recovery using Additive Energy) Suppose $f : \mathbb{Z}_N \to \mathbb{C}$ is a non-zero signal with auto-correlation F supported on A. Suppose that W_F is supported on a set with Sidon structure. Define*

$$E_\Delta(A) = \{(u, v), (a, b) \in A \mid u + v = a + b\}$$

Then

$$\|G\|_2^4 \cdot N \leq 2|A| \cdot |E_\Delta(A)| \tag{17}$$

Where $G = \frac{F}{\|F\|_\infty}$ *and* $\|F\|_\infty$ *is the maximum value attained by the function (since domain is discrete).*

Proof. Many of the step are readily available to us. From the proof of the previous theorem, it immediately follows that

$$\|F\|_4^4 \leq 2N^{-1} \left(\sum_{y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^2 \right)^2$$

Now we can use a trick to get something that looks like equation 16.

$$\sum_{y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^2 \leq \|F\|_\infty^2 \sum_{y \in \mathbb{Z}_N} 1_A\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \|F\|_\infty^2 \cdot g'(x)$$

But in this case

$$g'(x) = |\{(u, v) \in A \mid u + v = 2x\}|$$

Notice this is different from equation 16 because our indicator function does not necessarily split nicely here. However it is still worth looking at since

$$\sum_{x \in \mathbb{Z}_N} [g'(x)]^2 = |\{(u, v), (a, b) \in A \mid u + v = a + b\}| = |E_\Delta(A)|$$

Since $E_\Delta(A)$ is defined in terms of tuples in A, it is no longer the standard additive energy. Then

$$2N^{-1} \left(\sum_{y \in \mathbb{Z}_N} \left| F\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \right|^2 \right)^2 \leq 2N^{-1} \|F\|_\infty^4 |E_\Delta(A)|$$

Now putting this together with the original inequality it follows that

$$\left(\frac{\|F\|_4}{\|F\|_\infty} \right)^4 \leq 2N^{-1} |E_\Delta(A)|$$

Let $G = F / \|F\|_\infty$, then from Cauchy-Schwarz [2]

$$\|G\|_2^4 \leq \|G\|_4^4 \cdot |A|$$

Hence

$$\|G\|_2^4 \leq 2N^{-1}|A| \cdot |E_\Delta(A)|$$

Rearranging gives us our result □

Remark. Let F be an auto-correlation, such that

$$\left\{F\left(x+\frac{y}{2}, x-\frac{y}{2}\right)\right\}_{(x,y)\in S} \text{ is missing}$$

Assume that W_F is supported on a Sidon set. Then if F cannot be uniquely recovered, there must exists two possible autocorrelation F_1, F_2 such that F_1, F_2 match F for $(x+y/2, x-y/2) \notin S$. Define

$$H = F_1 - F_2$$

then by linearity of the Fourier-Wigner transform $W_H = W_{F_1} - W_{F_2}$, so W_H is supported on a Sidon set as well. Define $G_H = \frac{H}{\|H\|_\infty}$, then by the uncertainty estimate

$$\frac{N}{2} \cdot \|G_H\|_2^4 \leq 2|S| \cdot |E_\Delta(S)|$$

If G_H satisfies $\|G_H\|_2^2 \geq c|S|$ (which is certainly valid in the case where $|H|$ maps to $\{0,1\}$, in which case this holds with $c = 1$). Then if the set S satisfies

$$\frac{c^2}{2} N \cdot |S| > |E_\Delta(S)|$$

it follows that $H = 0$, by the uncertainty principle and F can be uniquely recovered.

While this construction relies on knowledge of $|E_\Delta(S)|$, since the set S is known this quantity can be computed out explicitly in each case.

4 Further works

As a final remark on the previous result, one natural direction for improvement is to refine the bounds using more precise information about the additive structure of A. In particular, stronger control on the additive energy $E_\Delta(A)$ may yield sharper quantitative constraints on the size and structure of A. Notably, the size of the set supporting W_F does not explicitly appear in our estimates, suggesting that there may be nontrivial ways to incorporate this parameter into a more complete uncertainty principle.

It would also be of interest to compare the behavior of the Fourier–Wigner L^4 norm with other measures of additive structure, in order to better understand the extent to which phase-space representations capture combinatorial properties of sets.

More broadly, we have shown that the Fourier–Wigner transform exhibits a structure that parallels that of the Fourier transform in several key respects. A systematic comparison between the two transforms in the context of signal recovery could provide further insight, particularly in identifying regimes where the Fourier–Wigner transform offers advantages over classical Fourier-based methods. Additionally, it may be fruitful to exploit properties unique to the Fourier–Wigner transform, such as the projection property, to establish more direct connections with Fourier analytic techniques and potentially derive new recovery results.

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