Math 467: Complex Analysis

Alex Iosevich

Final Exam (Spring 2024)

NAME (please print legibly): ____________________

Your University ID Number: ____________________

Your University email ____________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____________________

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **You are responsible for checking that this exam has all 27 pages.**

Problem #1. Compute the following integrals. You must state (not prove) any theorems you wish to use in the process.

i) Compute the following integral using the calculus of residues. Explicitly justify every step.

$$\int_{-\infty}^{\infty} \frac{x-1}{x^5-1} dx.$$

ii) Compute the following integral using the calculus of residues. Explicitly justify every step.

$$\int_{0}^{\infty} \frac{\sin^2(x)}{x^2} dx.$$

Hint: You may want to consider expressing $\sin^2(x)$ in terms of $\cos(2x)$ and then expressing $\cos(2x)$ using complex exponentials.

Problem #2. i) Suppose γ is a closed curve in region Ω with $\gamma \sim 0$ in Ω and $n(\gamma, z) = 0$ or 1 for all $z \in \Omega \setminus \gamma$. Prove that if f and g are analytic in Ω and satisfy

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

for all $z \in \gamma$, then f and g have the same number of zeroes enclosed by γ.

ii) Does the function $e^z - z$ have any zeroes on the unit disk centered at the origin? Justify your assertion.

Hint: Consider a suitable function on the unit disk that has no zeroes.

Problem #3. i) Suppose f is analytic in the open unit disk $\mathbb{D}$ and suppose $|f(z)| < 1$. If $z, a \in \mathbb{D}$, then

$$\left|\frac{f(z) - f(a)}{1 - \overline{f(a)}f(z)}\right| \leq \left|\frac{z - a}{1 - \overline{a}z}\right|$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

If you use the maximum principle in your argument, you must state it precisely and prove it.

ii) Suppose f is analytic in the open unit disk $\mathbb{D}$ and suppose $|f(z)| < 1$. Is it possible for such a function to satisfy $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$? If such a function exists, construct it. If it does not exist, prove it. Justify all your work.

Problem #4. i) Prove that if f is analytic on $\mathbb{C} \setminus \{z_0\}$ for some $z_0 \in \mathbb{C}$ and one-to-one, then

$$f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$.

ii) Let Ω be the intersection of the disk of radius 1 centered at the origin and the disk of radius $\sqrt{2}$ centered at $1+i$. Construct a conformal map that maps Ω to the unit disk centered at the origin.

Problem #5. Let f_n be analytic over a region Ω in the complex plane. If there is an $M > 0$ such that

$$\int\int_{\Omega} |f_n(x + iy)|^2 dxdy \leq M$$

for each n, then $\{f_n\}$ is a normal family. Your solution must include a precise (and correct) definition of a normal family of functions.

Hint: On each compact subset X of Ω, prove the existence of $r > 0$ such that $B(a, r)$ is contained in Ω for every $a \in X$. Bound the displayed integral from below by the integral over $B(a, r)$ and use it to obtain an upper bound for $|f(a)|$.