Math 467: Theory of Analytic Functions

Final

May, 2023

NAME (please print legibly): ______________________________
Your University ID Number: ______________________________

Instructions:

1. Read the notes below:

   - **Using any notes, books, online resources, or contacting any other people during this exam is prohibited.**

2. Read the following Academic Honesty Statement and sign:

   I affirm that I will not use any unauthorized resourced, or give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

1. (10 points) If f is a harmonic function in a domain Ω, then show that f^2 cannot be harmonic unless f is a constant function.

2. (10 points) Let $a_1, a_2, \ldots$ be a sequence of distinct complex numbers, and $A_1, A_2, \ldots$ any sequence of complex numbers. Either prove that there must exist an entire function f with $f(a_n) = A_n$ for all $n \in \mathbb{N}$, or give a counterexample and justify it.

3. (10 points) Prove, that if f is an entire function which is injective, then $\exists a \neq 0, b \in \mathbb{C}$ such that $f(z) = az + b$.

Hint: f will have an isolated singularity at ∞. You can argue by analyzing the type of singularity f can have at ∞.

4. (10 points) Evaluate $\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx$ where t is any real constant.

5. (10 points)

a) Recall, that given a non-empty open subset G of $\mathbb{C}$, the Bergman space $L_a^2(G)$ is defined to be the space of analytic functions $f : G \to \mathbb{C}$ such that $\int_G |f|^2 dxdy < \infty$, and it is a Hilbert space with respect to the inner product $< f, g >:= \int_G f\bar{g}dxdy$.

Find an orthonormal basis of $L_a^2(D)$, where D is the unit disk.

b) Prove that if $\sum_{n=0}^{\infty} |a_n|^2$ converges, then $\sum_{n=0}^{\infty} \sqrt{n+1}a_n z^n$ is analytic in the region $|z| < 1$.

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