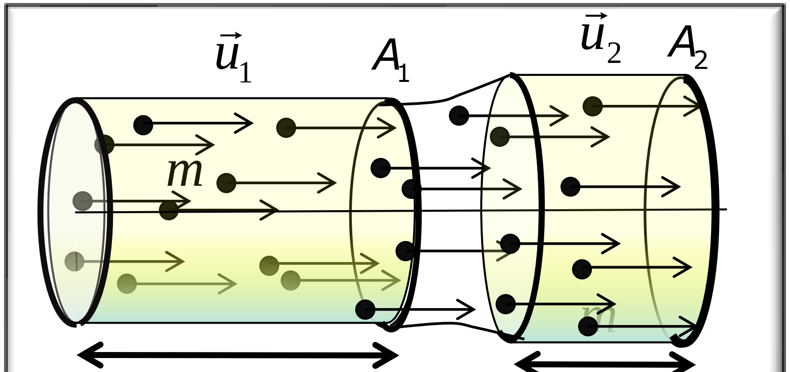
Fluid Dynamics Elements

Elements of Fluid Dynamics: Ideal Laws



$$\Delta x_1 = u_1 \cdot \Delta t \qquad \Delta x_2 = u_2 \cdot \Delta t$$

$$\Delta V_1 = \Delta x_1 \cdot A_1 \qquad \Delta V_2 = \Delta x_2 \cdot A_2$$

pressure p_1 pressure p_2

Ideal incompressible, non-viscous liquid, "streamlines" (irrotational flow, no inertia)

Number density : $\rho = const.\quad [\# N/cm^3]$

Mass density : $\rho_m = m \cdot \rho = const.\quad [g/cm^3]$

Flux (# particles / time) through tube x – scn area $A = A_\perp$

$$\dot{N} = \frac{dN}{dt} = \rho \cdot u \cdot A \;\rightarrow\; \frac{dm}{dt} = \rho_m \cdot \underbrace{(u \cdot A)}_{dV/dt} = \rho_m \cdot \dot{Q}$$

Incompressibility ($\rho_1 = \rho_2$) → equal # of particles (N) flow out of V_1 and into V_2.

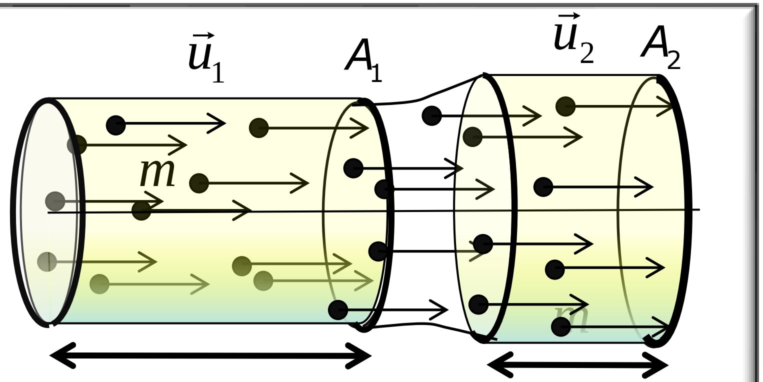
$$\dot{N}_1 = \rho \cdot u_1 \cdot A_1 = j_1 \cdot A_1 = \rho \cdot u_2 \cdot A_2 = j_2 \cdot A_2 = \dot{N}_2$$

Pushing particles from V_1 to V_2 requires work **Δw=** $\Delta(E_{kin}+V_{pot})$. Static pressure *p*.

Continuity Equation

$$j_m \cdot A = \rho_m \cdot u \cdot A = \frac{dm}{dt} = const.$$

Application of Fluid Dynamics: Ideal Laws



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Pushing particles from V_1 to V_2 requires work **$\Delta w = \Delta(E_{kin} + V_{pot})$**. Static pressure *p*.

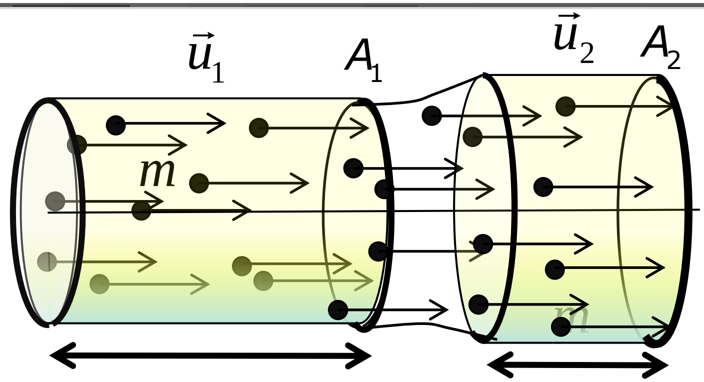
$$\Delta w_{1\to 2} = \left[(p_2 \cdot A_2) \cdot u_2 - (p_1 \cdot A_1) \cdot u_1\right] \Delta t = \frac{\Delta m}{\rho_m}(p_2 - p_1)$$

$$Loss(?)\;\; \Delta E_{kin} = (1/2) \cdot \rho_m \cdot \left(A_2 \cdot u_2^3 - A_1 \cdot u_1^3\right) \Delta t$$

$$Gain(?)\;\; \Delta V_{Pot} = \Delta m\left(\upsilon_{Pot,2} - \upsilon_{Pot,1}\right), \quad \Delta m = \rho_m \cdot A \cdot u \cdot \Delta t$$

$$\frac{\Delta m}{\rho_m}(p_1 - p_2) = (1/2)\Delta m\left(u_2^2 - u_1^2\right) + \Delta m\left(\upsilon_{Pot,2} - \upsilon_{Pot,1}\right)$$

Application of Fluid Dynamics: Ideal Laws



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Continuity Equation

$$j_m \cdot A = \rho_m \cdot u \cdot A = \frac{dm}{dt} = const.$$

Bernoulli Equation

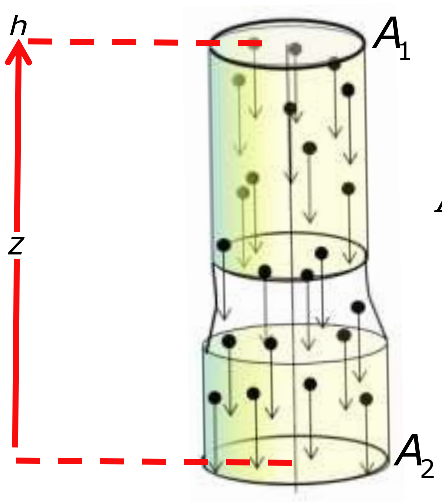
$$p + (1/2)\rho_m u^2 + \rho_m \upsilon_{Pot} = \frac{E}{V} = const.$$

Convective Energy Transfer

Apply *Bernoulli Equation* to flow of water with gravitational energy $\Delta V_{pot}=V_{pot}=m\cdot g\cdot h$

At $z=\mathbf{h}$: $p=0,\ u=0,\ V_{pot}=m\cdot g\cdot h$

$$\cancel{p} + \frac{1}{2}\rho_m \cancel{u^2} + \rho_m \upsilon_{Pot} = \rho_m \cdot g \cdot h$$



$$A_1 = A_2 = A$$

At bottom, $z=0$: $V_{pot}:=0$

$p \le \rho_m\cdot g\cdot h$ ($=$ for $u=0$)

$$p + \frac{1}{2}\rho_m u^2 + \rho_m \cancel{\upsilon_{Pot}} = \rho_m \cdot g \cdot h$$

Static backup pressure

In free fall through potential difference $\Delta V_{pot}=m\cdot g\cdot h$, no static "backup" pressure differential $(p=0)$ → ***"jet"***

$$E_{kin}\ per\ \Delta V \rightarrow (1/2)\,\rho_m u^2 = \rho_m \cdot g \cdot h \rightarrow \quad u = \sqrt{2g\cdot h}$$

If stream with velocity u exits through area A, →

Volume flow rate $\dot{Q} := \dot{V} = dV/dt$ and power $P =$

$$\dot{Q} = \frac{dV}{dt} = A\cdot u = A\cdot\sqrt{2\cdot g\cdot h} \qquad [\,] = Volume/Time$$

$$P = \underbrace{\dot{Q}\cdot\left(\rho_m\right.}_{dm/dt}\left.\cdot g\cdot h\right) \approx 45\cdot A\cdot h^{3/2}\ kW$$

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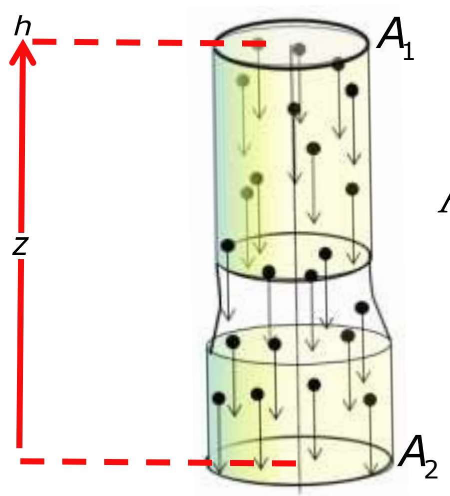

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$$P = \underbrace{\dot{Q}\cdot\left(\rho_m\right.}_{dm/dt}\left.\cdot g\cdot h\right) \approx 45\cdot A\cdot h^{3/2}\ kW$$ Rule of Thumb

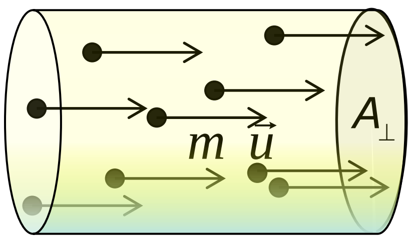
Example: Head at $h=175\ m$, diameter of penstock $d=3\ m$ ($A = \pi\cdot(d/2)^2=2.41\text{m}^2$)

$$u = \sqrt{2\cdot g\cdot h} = \sqrt{2\cdot 9.81\cdot 175}\,\frac{m}{s} = 58.6\,\frac{m}{s}$$

$$\dot{Q} = A\cdot u = 2.41\,m^2\cdot 58.6\,\frac{m}{s} = 141.3\,\frac{m^3}{s} = 1.4\cdot 10^5\ L/s$$

$$\rightarrow \boxed{P = 251\ MW\ (= P_{max})\quad contained\ in\ flow}$$

Fluid Resistance/Parasitic Drag



$\Delta x = u \cdot \Delta t$

$\Delta V = A_\perp \cdot \Delta x$

Estimation of parasitic drag, angle of particle flow component perpendicular to obstacle area.

Continuous flow of particles (mass m),

Number density $\rho\ [\#/cm^3]$, *mass density* $\rho_m = m \cdot \rho$

Mass flux density : $j_m(u) = m \cdot \underbrace{\rho \cdot u}_{j(u)} = \rho_m \cdot u\ [g/cm^2 \cdot \Delta t]$

Flow speed $u = u_\perp$ ($\perp$ *to area* A) $\rightarrow$ *Kinetic energy density* $e_{kin} = \frac{1}{2} m \cdot \rho \cdot u^2$

Energy flux per Δt *onto area* $A = A_\perp$ ($\perp$ *to wind direction*) :

$$\Delta E = \frac{1}{2}(m \cdot \rho \cdot u^2) \cdot \underbrace{(A \cdot u \cdot \Delta t)}_{\Delta V} \rightarrow \text{Power } P = \frac{1}{2} A \cdot m \cdot \rho \cdot u^3 \rightarrow \boxed{P = \frac{1}{2} A \cdot \rho_m \cdot u^3}$$

Get force exerted on area A from : $P = F \cdot u \rightarrow F =: F_{drag}$

$$\boxed{F_{drag} = \frac{1}{2} A \cdot \rho_m \cdot u^2}$$

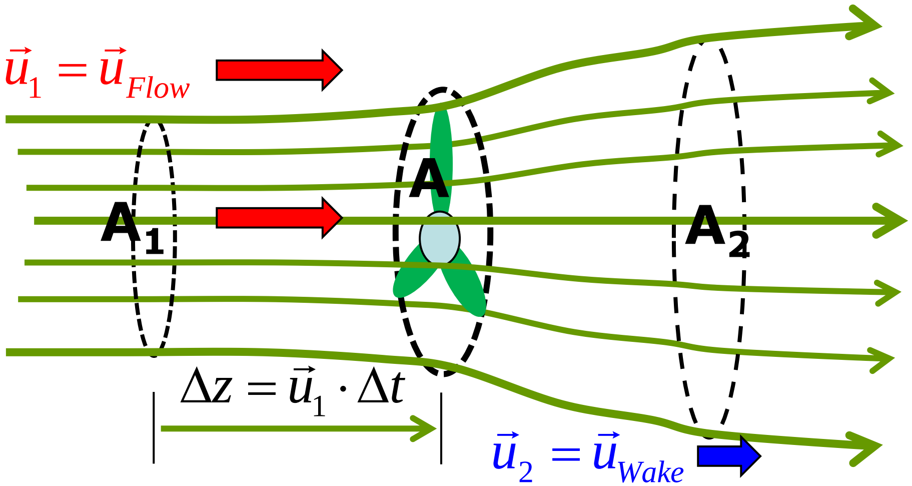
Effective (projected) area hit directly by wind : $A_\perp = C_d \cdot A_{total}$

$$F_{drag} = D = \frac{1}{2} C_d \cdot A_{total} \cdot \rho_m \cdot u^2$$

$$F_{lift} = L = \frac{1}{2} C_L \cdot A_{total} \cdot \rho_m \cdot u^2$$

Derivation valid for parasitic drag, e.g., air resistance. Often, experimentally determined **Drag Coefficients** represent total drag/resistance.

Fluid-Dynamic Power Transfer



At obstacle (turbine), u slows, stream-lines diverge, flow speed decreases, $\mathbf{u_2 = u_{wake} < u_1 = u_{wind}}$

$$E_{kin} = \frac{1}{2}\cdot\left(\rho_m \cdot \Delta V\right)\cdot u_1^2, \quad volume\ \Delta V$$

$$through\ A\ in\ \Delta t:$$

$$Volume\ \Delta V\left(u_1\right) = A\cdot u_1 \cdot \Delta t.$$

$$Power\ \ flux \perp A: \quad P_i = \frac{\Delta E_{kin}}{\Delta t} = \frac{1}{2}\cdot\left(\rho_m \cdot A_i \cdot u_i^{3}\right)$$

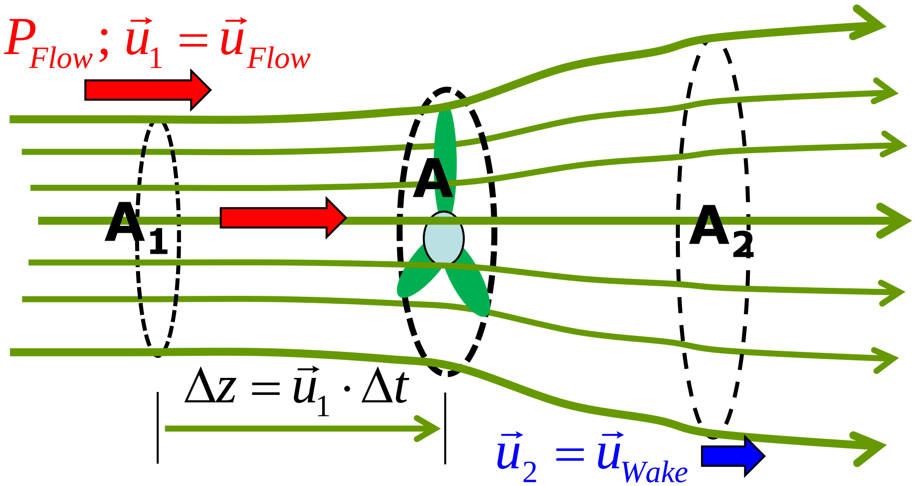
$$Continuity: j_1 A_1 = \rho A_1 \cdot u_1 \approx \rho A_2 \cdot u_2 = j_2 A_2$$

$$\rightarrow Average\ speed\ \bar{u} := \left(u_1 + u_2\right)/2\ \ for\ mass\ \ flow\ \dot{M} = \rho_m \cdot \Delta V/\Delta t = \rho_m A \bar{u}$$

$$Volume\ \Delta V\left(\bar{u}\right)\ transfers\ to\ turbine$$

$$\Delta P = P_1 - P_2 \approx \frac{\left(\rho_m A \bar{u}\right)}{2}\left(u_1^2 - u_2^2\right) = \frac{\left(\rho_m A\right)}{4}\left(u_1 + u_2\right)\left(u_1^2 - u_2^2\right)$$

Fluid-Dynamic Power Transfer



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through A in Δt:

$$Volume\ \Delta V(u_1) = A \cdot u_1 \cdot \Delta t.$$

Delivered to turbine: $\Delta P =: C_{Turbine} P_{wind}$ *defines* *power coefficient* $C_{Turbine} \rightarrow$

$$C_{Turbine} \approx \frac{1}{2u_1^3} \cdot (u_1 + u_2) \cdot (u_1^2 - u_2^2) = \frac{1}{2} \cdot (1+x) \cdot (1-x^2) \quad with\ \ x := \frac{u_2}{u_1}$$

$$Maximum\ power: d(\Delta P)/dx = 0 \rightarrow x\big|_{\Delta P = \max} = 1/3 \rightarrow self\ \ regulating\ stable$$

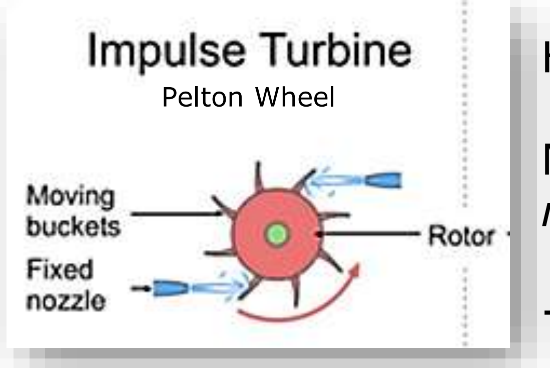
$$Effective\ mean\ speed\ \bar{u} := \frac{1}{2} u_1 (1+x) = \frac{2}{3} u_1$$

$$C_{Turbine} = \frac{\Delta P}{P_{Wind}} \leq \frac{16}{27} = 0.593 \quad Betz\ Limit$$

$$\bar{u} := (1-a) u_{Wind}$$

a = *linear (axial) induction factor of turbine* $= f(\#blades, A_i)$

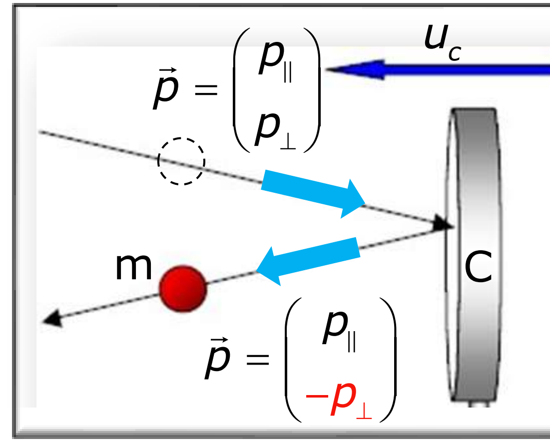
Power Generation in Impulse Turbines



Heads > 300m @ atm. P

Momentum transfer $\Delta p_{\perp}$ by m colliding with bucket (C),

→ L-transfer $\Delta\vec{L} = \vec{r} \times \Delta\vec{p}_{\perp}$



Mass (H_2O) *flow density* : $j_m(u) = \rho_m \cdot u \left[\dfrac{g}{cm^2 \cdot \Delta t}\right]$

Bucket area A_c *is hit per* Δt *by (velocity jet* u, *bucket* u_c *)*

$\Delta m \approx j_m(u - u_c) \cdot A_c \cdot \Delta t$, *rel. momentum* $p = \Delta m \cdot (u - u_c)$,

$(u - u_c)$ = *speed relative to bucket*
initally $u_c \ll u$, *increases in time.*

$\rightarrow$ *transfer* $\Delta p = 2p$ *to cup in* $\Delta t \rightarrow$ *Force* $F = \Delta p / \Delta t$

$\rightarrow$ *Force* $F = 2p/\Delta t \approx 2 \cdot \left[j_m (u - u_c) \cdot A_c \right] \cdot (u - u_c) = 2\rho_m A_c (u - u_c)^2$ ← Example of dissipative force

Energy (work) transfer to bucket : $\Delta E = F \cdot \Delta x = F \cdot u_c \cdot \Delta t$

Power transferred to bucket : $P = F \cdot u_c = \left[2\rho_m A_c\right](u - u_c)^2 \cdot u_c$

Speed of bucket increases, maximum power transfer $P_{max} \rightarrow u_c = \dfrac{1}{3} u$

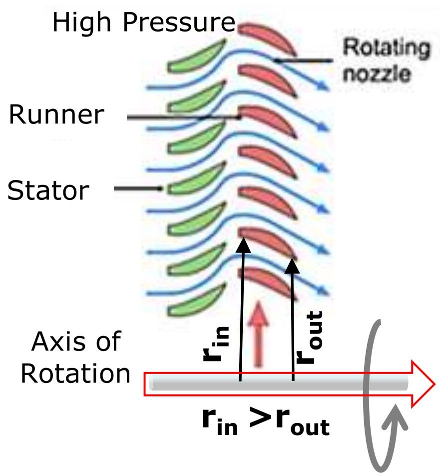
$$P_{max} = \frac{2}{27}\left[\rho_m \cdot A_c\right] \cdot u^3 = \frac{4}{27}\left[\frac{1}{2}\rho_m \cdot u^2\right] \cdot \underbrace{(A_c \cdot u)}_{\dot{Q} = dV/dt} = \frac{4}{27}\frac{d}{dt}\left(\frac{1}{2} m \cdot u^2\right)$$

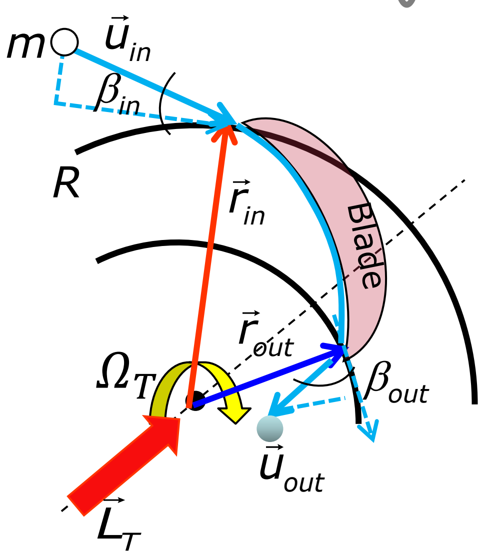
Converts $\lesssim$16% of incoming P

Other Turbines

Angular Momentum Transfer in Reaction Turbines

Gen Reaction Turbine

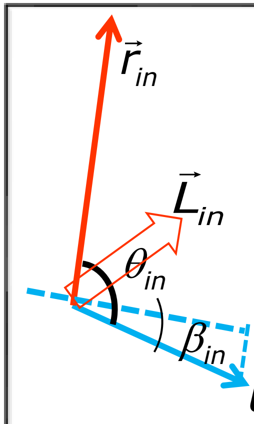




Energy transfer leads to speed up of turbine rotation = increased angular momentum $\vec{L}_T$ by $\Delta\vec{L}_T$. Driving fluid has lost this angular momentum.

$$\vec{L}_{out} = \vec{L}_{in} - \Delta\vec{L}_T$$

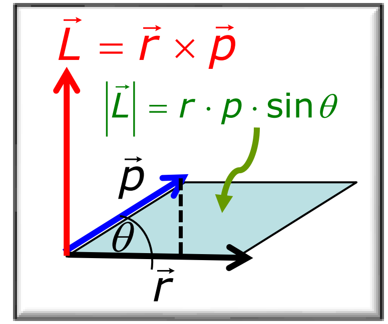
all parallel



Incoming mass

$$u_{in} \cdot \sin\theta_{in} = u_{in} \cdot \cos\beta_{in}$$

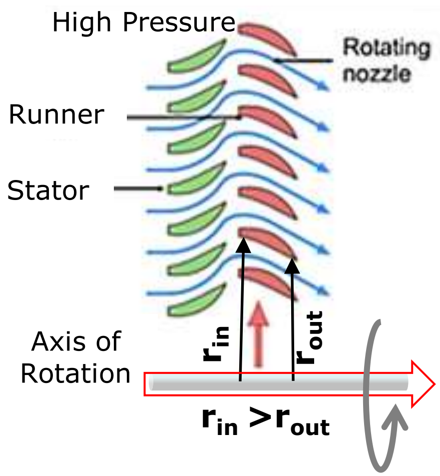
$$L_{in} = m \cdot r_{in} \cdot u_{in} \cdot \cos\beta_{in}$$

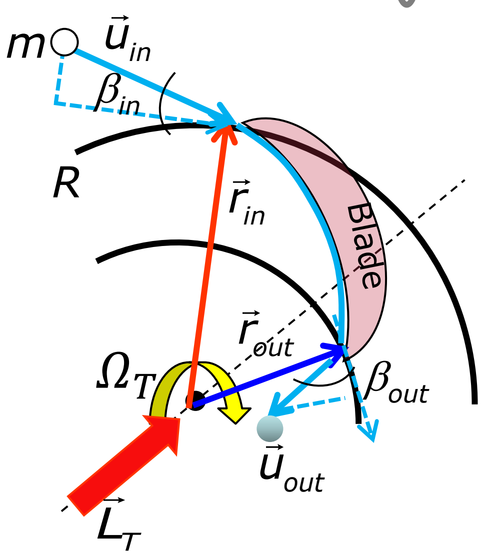


Outer vector product of two vectors $\vec{r}$ and $\vec{p}$

Angular Momentum Transfer in Turbines

Gen Reaction Turbine





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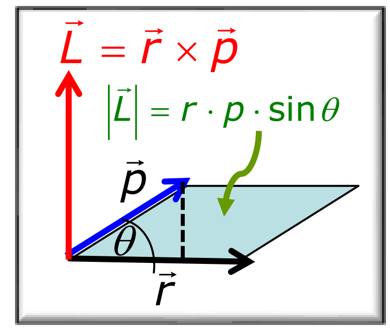

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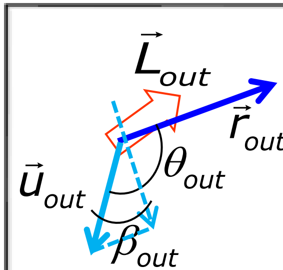


Outgoing mass

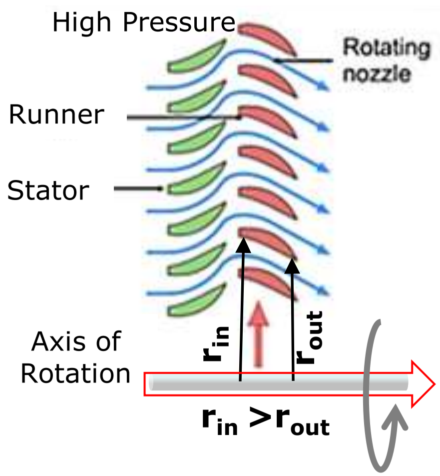
$$u_{out} \cdot \sin\theta_{out} = u_{out} \cdot \cos\beta_{out}$$

$$L_{out} = m \cdot r_{out} \cdot u_{out} \cdot \cos\beta_{out}$$

$$\Delta L_T = L_{in} - L_{out}$$

Angular Momentum Transfer in Hydro Turbines

Gen Reaction Turbine



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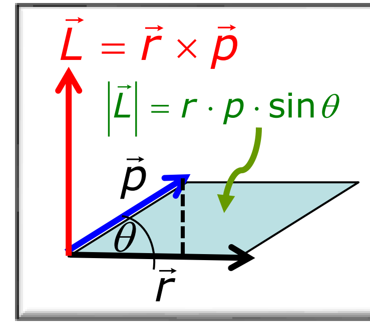
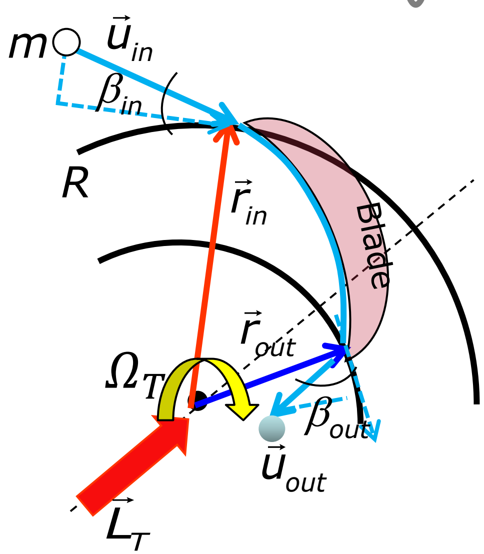
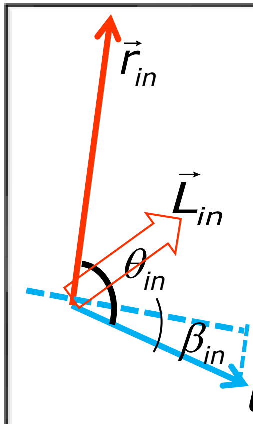
$$\vec{L}_{out} = \vec{L}_{in} - \Delta\vec{L}_T$$ all parallel



Outer vector product of two vectors $\vec{r}$ and $\vec{p}$





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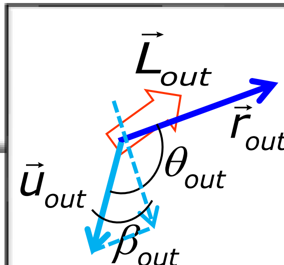

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$$L_{out} = r_{out} \cdot \underbrace{(m \cdot u_{out})}_{p_{out}} \cdot \cos\beta_{out}$$

Transferred : $\Delta L_T = L_{in} - L_{out}$

$$\rightarrow \textit{Torque } M = \Delta L_T / \Delta t = \dot{m} \cdot (r_{in} \cdot u_{in} \cdot \cos\beta_{in} - r_{out} \cdot u_{out} \cdot \cos\beta_{out})$$

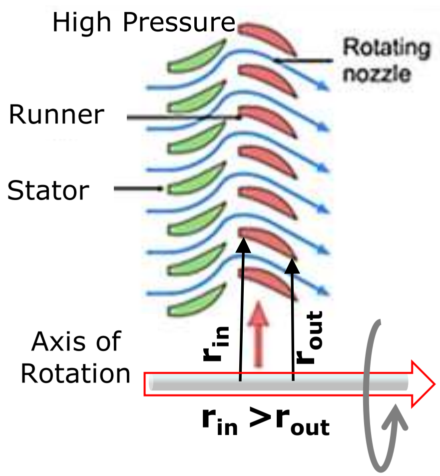
Power $P = M \cdot \Omega_T$; *Mass flow* $\dot{m} = \rho_m \cdot \dot{Q}$

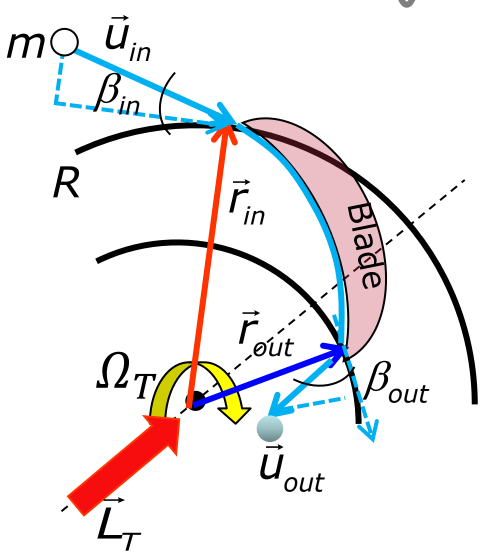
Euler's Turbine Equation

$$P = \Omega_T \cdot \rho_m \cdot \dot{Q} \cdot (r_{in} \cdot u_{in} \cdot \cos\beta_{in} - r_{out} \cdot u_{out} \cdot \cos\beta_{out})$$

Angular Momentum Transfer in Turbines

Gen Reaction Turbine





Angular momentum to turbine (runner) by driving fluid (water)

$$\Delta\vec{L}_T = \vec{L}_{in} - \vec{L}_{out}\text{; } Torque\ M = \Delta L_T/\Delta t\text{ ; } Power\ P = M\cdot\Omega_T$$

Turbine power does not depend on many construction details but blade geometry → maximize angular momentum transfer!

$$P = \Omega_T \cdot \rho_m \cdot \dot{Q}\cdot\left[\left(\vec{r}\times\vec{u}\right)_{in} - \left(\vec{r}\times\vec{u}\right)_{out}\right]$$

Euler's Turbine Equation

Power is maximized if fluid brings in maximum angular momentum ($\beta_{in}=0^0$) and carries no angular momentum on the way out ($\beta_{out}=90^0$) → tangential inflow & radial outflow.

$$P_{max} = \Omega_T \cdot \underbrace{\rho_m \cdot \dot{Q}}_{=dm/dt} \cdot r_{in} \cdot u_{in} \cdot \cos\beta_{in} \quad \rightarrow$$

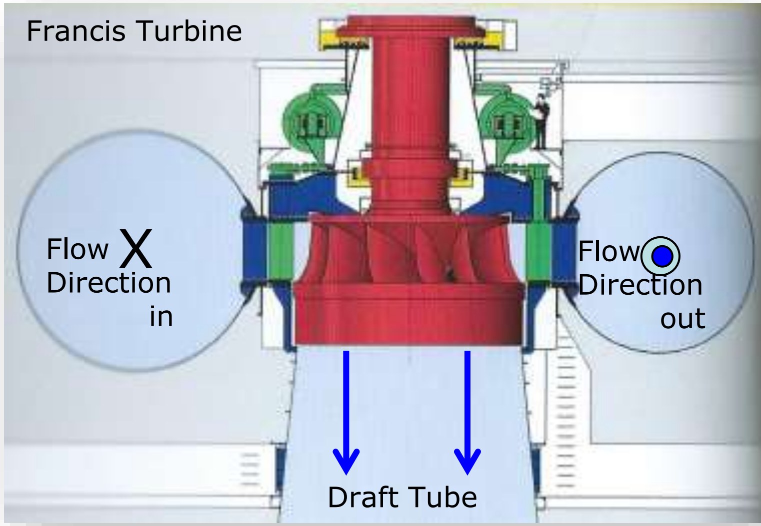
$$P_{max} = \dot{m}\cdot R \cdot \Omega_T \cdot \left(u_{in}\right)_{tang}$$

$R =$ *injection radius for turbine,*
$\left(u_{in}\right)_{tang} =$ *tangential jet velocity*

High power produced by large turbines: high water inflow + tangential injection ($\beta_{in}=0^0$) + radial outflow ($\beta_{out}=90^0$).

Synchronized el. power output steered by governor circuitry controlling gate position.

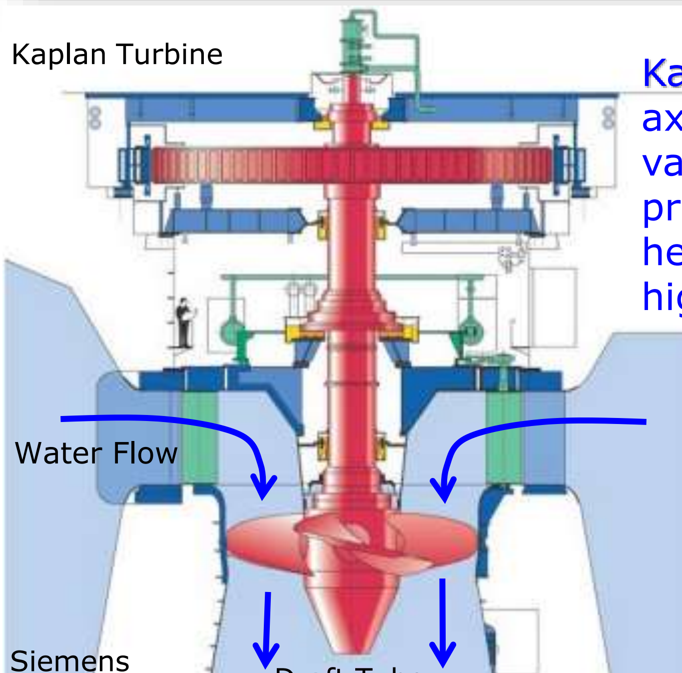
Turbine Types



Hydro turbines are impact or reaction turbines.

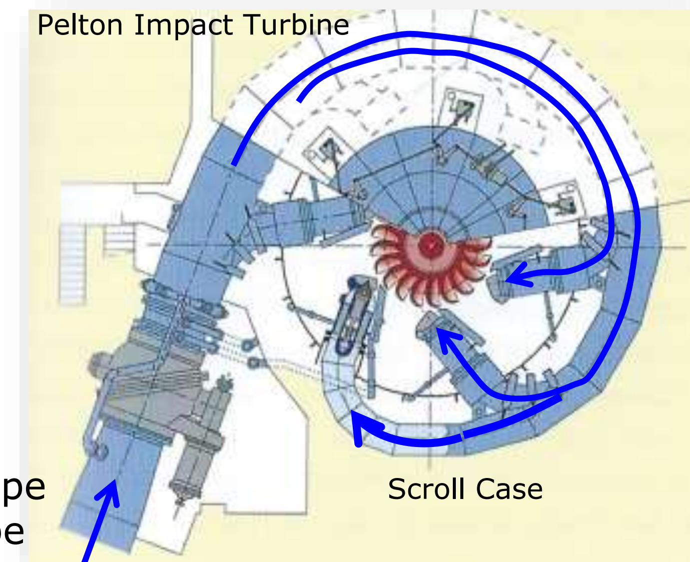
Francis Turbine, radial flow, dia 0.5- 6 m
Fully submerged, horizontal or vertical modes.
Axial outflow.

Popular design, versatile & useful for very different effective heads



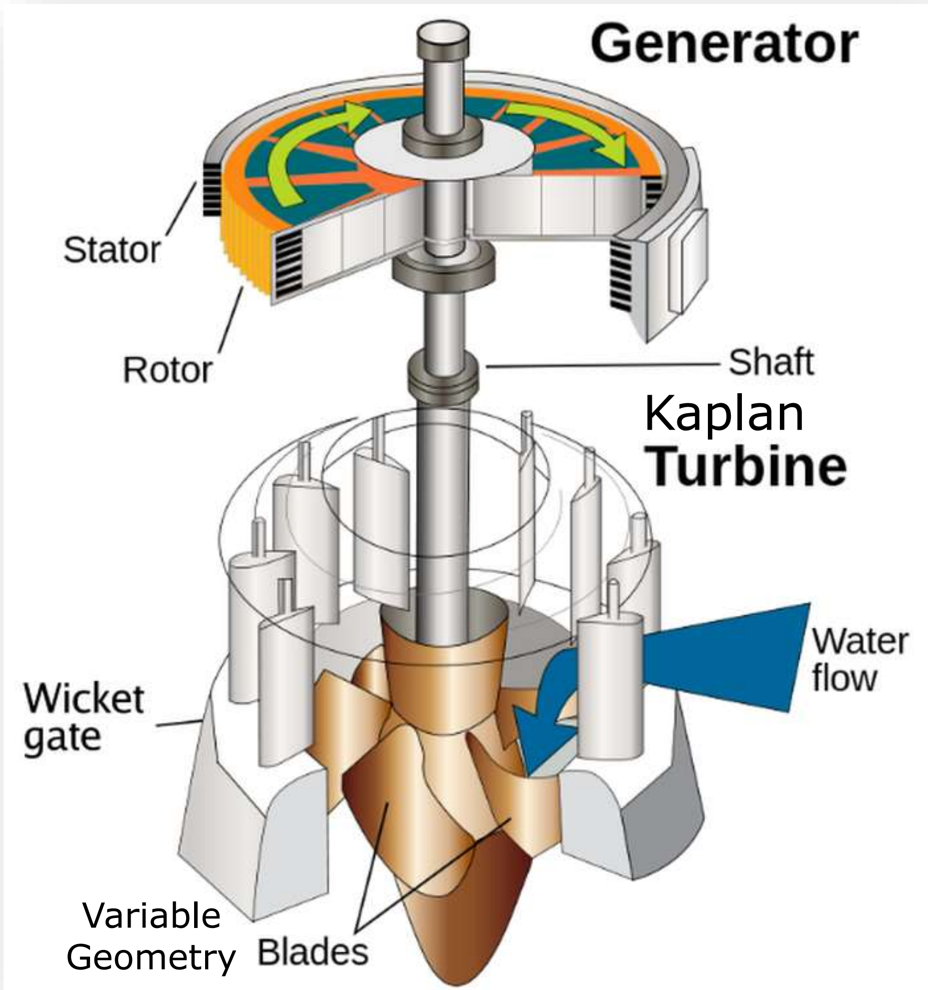
Kaplan Turbine, axial flow, variable-pitch propeller. Low head (<50m), high flow.

Pelton impact/impulse turbine, tangential flow, fixed buckets, low head, low/medium flow.



Scroll type inlet tube

Turbine Arrangements



Wikipedia

Propeller turbines for low heads. Fixed blades or variable pitch. Schematics of power generation with a Kaplan turbine = high efficiency @all loads/heads because of adjustable propeller blades.

Francis turbine: Heads< 360 m. Guide vanes→ tangential injection → radial out flow
"Runner"