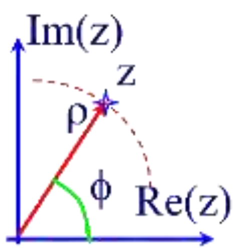
Complex Numbers

Representations of Complex Numbers



There are two ways of representing a complex number as a point in the Complex Plane, in Cartesian or polar coordiantes. The ***Real*** and ***Imaginary*** parts of a complex number *z* are the components (like*x* and *y*) of the number *z* in a Cartesian coordinate system. One writes

$$z = x+i\cdot y = Re\{z\}+i\cdot Im\{z\} \qquad (1)$$

Here,

$$i := \sqrt{(-1)} \qquad (2)$$

is the unit of the ***imaginary numbers***. The quantity

$$|z| = \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} \qquad (3)$$

is the norm or absolute value of z.

The number *z* has a ***complex conjugate*** number ***z****, which is obtained from *z* by changing the sign of the imaginary part:

$$z^* = (\mathrm{Re}\, z + i\cdot \mathrm{Im}\, z)^* = (\mathrm{Re}\, z - i\cdot \mathrm{Im}\, z) \qquad (4)$$

It is easy to see that, because of $(a+b)\cdot(a-b) = a^2 - b^2$,

$$\begin{aligned} |z| &= \sqrt{(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2} = \sqrt{(\mathrm{Re}\, z)^2 - (i\cdot \mathrm{Im}\, z)^2} \\ &= \sqrt{(\mathrm{Re}\, z + i\cdot \mathrm{Im}\, z)\cdot(\mathrm{Re}\, z - i\cdot \mathrm{Im}\, z)} = \sqrt{z\cdot z^*} \end{aligned} \qquad (5)$$

Instead of using the above Cartesian coordinates, the point z can also be represented by the length $\rho \geq 0$ of a ray from the origin to the number and the "phase angle" ϕ which the ray encloses with the real axis. Then,

$$z = \rho e^{i\phi} \tag{6}$$

The equivalence of the representations (1) and (6) requires that

$$\rho = |z| \tag{7}$$

and

$$\phi = \arctan\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right) \tag{8}$$

Since

$$\operatorname{Re} z = |z| \cdot \cos(\phi) \quad \text{and} \quad \operatorname{Im} z = |z| \cdot \sin(\phi) \tag{9}$$

one also derives ***Euler's Formula***

$$e^{i\phi} = \cos(\phi) + i \cdot \sin(\phi) \tag{10}$$

Calculations with Complex Numbers

Addition and subtraction of two complex numbers, z_1 and z_2 are best performed in the Cartesian representation:

$$\begin{aligned} z = z_1 + z_2 &= \operatorname{Re}(z_1 + z_2) + i \cdot \operatorname{Im}(z_1 + z_2) \\ &= \{\operatorname{Re}(z_1) + \operatorname{Re}(z_2)\} + i \cdot \{\operatorname{Im}(z_1) + \operatorname{Im}(z_2)\} \end{aligned} \tag{11}$$

Multiplication with a real number α or another complex number is best performed in the polar representation of Equ. 6:

For example,

$$\alpha z = (\alpha \rho) e^{i\phi} \tag{12}$$

and

$$\begin{aligned} z = \rho e^{i\phi} &= z_1 \cdot z_2 = \left(\rho_1 e^{i\phi_1}\right)\cdot\left(\rho_2 e^{i\phi_2}\right) = \\ &= \left(\rho_1 \cdot \rho_2\right)\cdot\left\{e^{i\phi_1}\cdot e^{i\phi_2}\right\} = \left(\rho_1\cdot\rho_2\right)\cdot e^{i\left(\phi_2+\phi_1\right)} \end{aligned} \tag{13}$$

In other words, in a multiplication of two numbers, their absolute values multiply, while their phases add. The absolute value of a "phase factor" with $\rho = 1$ is

$$|z| = |e^{i\phi}| = \sqrt{\left(\mathrm{Re}\, z\right)^2 + \left(\mathrm{Im} z\right)^2} = \sqrt{\left(\cos(\phi)\right)^2 + \left(\sin(\phi)\right)^2} = 1 \tag{14}$$