A discussion of bases in Banach spaces and some of their properties

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Abstract

In this paper, we focus on the subject of Schauder bases or simply bases for infinite-dimensional Banach spaces and discuss various examples of such bases as well as several important generic properties and characterizations. These are based on a collection of results contained in an article written by Robert C. James [3].

1 Introduction

One of the important topics covered in an introductory linear algebra course is the concept of real/complex vector space and the related notion of dimension, which is almost exclusively discussed in the finite-dimensional setting. The only exception are usually the vector space of polynomials for which

$$\{1, x, x^2, \dots, x^n, \dots\}$$

is indicated as a basis and the vector space of real or complex-valued continuous functions defined on an interval of $\mathbb{R}$ for which the previous set is argued to be linearly independent.

One of the goals of this paper is to look at particular infinite-dimensional vector spaces and their bases, which are the Banach spaces and the related Schauder bases. In this direction, we discuss a number of well-known, important Banach spaces, for which we specify precise bases. The second goal here is to present several characterizations and properties for generic Schauder bases, which allow us to cover bases for other important functional, Banach spaces.

We start by recalling the classical terminology (e.g., definitions, notations) associated to Banach spaces, as well as important well-known results which are used later in the presentation. Next, we discuss several examples of canonical Banach spaces (e.g., Hilbert spaces, c_0, l^p with $1 \leq p < \infty$) and their bases, followed by a worked out nontrivial example of a basis for $C([0,1])$ which relies upon dyadic points. Following this, we present an abstract characterization theorem for Schauder bases which involves a very specific inequality. As an application of this result, we show that the Haar system of functions forms a basis for the Banach space $L^p([0,1])$, for each $1 \leq p < \infty$. Finally, we discuss a result which states that a sequence of points in a Banach space which is "reasonably close" to a basis is itself a basis.

2 Preliminaries

First, we remember that:

Definition 2.1. X is a vector space over a field of scalars $F = \mathbb{R}$ or $\mathbb{C}$ if one has two operations

$$+ : X \times X \to X \qquad \text{and} \qquad \cdot : F \times X \to X$$

such that $(X, +)$ is a commutative group and

$$\begin{aligned} &(a \cdot b) \cdot v = a \cdot (b \cdot v), \quad 1 \cdot v = v, \qquad (\forall)\, a, b \in F,\ v \in X, \\ &a \cdot (u + v) = a \cdot u + a \cdot v, \quad (a + b) \cdot v = a \cdot v + b \cdot v, \qquad (\forall)\, a, b \in F,\ u, v \in X. \end{aligned}$$

Based on this concept, we introduce:

Definition 2.2. X is a Banach space if X is a vector space with

$$\| \cdot \| : X \to \mathbb{R}_+$$

satisfying $\|v\| = 0$ iff $v = 0$,

$$\|av\| = |a|\|v\|, \quad \|u + v\| \leq \|u\| + \|v\|, \qquad (\forall)\, a \in F,\ u, v \in X,$$

and (X, d) is a complete metric space with

$$d(u, v) := \|u - v\|, \qquad (\forall)\, u, v \in X.$$

An important category of Banach spaces are the Hilbert spaces.

Definition 2.3. X is a Hilbert space if X is a Banach space with a inner product

$$\langle \cdot, \cdot \rangle : X \times X \to F$$

satisfying

$$\begin{aligned} &\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle, \qquad (\forall)\, u_1, u_2, v \in X, \\ &\langle au, v \rangle = a \langle u, v \rangle, \qquad (\forall)\, a \in F,\ u, v \in X, \\ &\langle v, u \rangle = \overline{\langle u, v \rangle}, \qquad (\forall)\, u, v \in X, \\ &\langle u, u \rangle = \|u\|^2, \qquad (\forall)\, u \in X. \end{aligned}$$

Other important examples of Banach spaces are as follows.

Example 2.1. $X = c_0$ is the vector space of all real sequences $u = \{u_n\}_{n\geq 1}$ for which

$$\lim_{n\to\infty} u_n = 0$$

and

$$\|u\| = \max_{n\geq 1}\{|u_n|\}.$$

Example 2.2. $X = l^p$ with $1 \leq p < \infty$ is the vector space of all real sequences $u = \{u_n\}_{n\geq 1}$ for which

$$\sum_{n\geq 1} |u_n|^p < \infty$$

and

$$\|u\| = \left(\sum_{n\geq 1} |u_n|^p\right)^{1/p}.$$

Example 2.3. $X = C([a,b])$ with $a < b \in \mathbb{R}$ is the vector space of continuous functions $f : [a,b] \to \mathbb{R}$ with

$$\|f\| = \max_{x\in[a,b]} |f(x)|.$$

It is easily seen that the convergence in norm coincides with the uniform convergence of functions on $[a,b]$.

Example 2.4. By slight abuse, we let $X = L^p([a,b])$ with $1 \leq p < \infty$ and $a < b \in \mathbb{R}$ be the vector space of Lebesgue measurable functions $f : [a,b] \to \overline{\mathbb{R}} = [-\infty, \infty]$ for which

$$\int_a^b |f(x)|^p \, d\lambda(x) < \infty,$$

where λ is the restriction of Lebesgue measure to $[a,b]$ and

$$\|f\| = \left(\int_a^b |f(x)|^p \, d\lambda(x)\right)^{1/p}.$$

At some point, we will use the following fundamental result for mappings between Banach spaces.

Theorem 2.1 (Inverse mapping theorem)**.** Let $T : X \to Y$ be a continuous, bijective linear mapping; i.e.,

$$T(u+v) = T(u) + T(v), \quad T(av) = aT(v), \qquad (\forall)\, a \in F,\ u, v \in X,$$

and

$$\|T\| := \sup_{\|u\|\leq 1} \|T(u)\| < \infty.$$

Then $T^{-1} : Y \to X$ is also continuous.

We end this section with another definition.

Definition 2.4. $a \in \mathbb{R}$ is called a dyadic point if

$$a = \frac{m}{2^n}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, and $n \geq 0$.

3 Canonical examples of Schauder bases

Here we are starting in earnest our discussion of bases in Banach spaces.

Definition 3.1. Let X be a Banach space. A sequence $\{e_n\}_{n\geq 1} \subset X$ is called a Schauder basis or simply a basis for X if for any $x \in X$ there exists a unique sequence $\{x_n\}_{n\geq 1} \subset \mathbb{R}$ such that

$$x = \sum_{n\geq 1} x_n e_n,$$

in the sense that

$$\lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} x_i e_i \right\| = 0.$$

The first example of a Schauder basis is in the context of Hilbert spaces.

Proposition 1. Let $\{e_n\}_{n\geq 1} \subset X$ be a complete orthonormal sequence in the Hilbert space X; i.e.,

$$\begin{aligned} &\langle e_n, e_m \rangle = \delta_{nm}, \qquad (\forall)\, n, m \geq 1, \\ &(\langle u, e_n \rangle = 0, \quad (\forall)\, n \geq 1 \implies u = 0)\,. \end{aligned}$$

Then $\{e_n\}_{n\geq 1}$ represents a basis for X.

Proof. First, we show that for any $x \in X$,

$$x = \sum_{n\geq 1} \langle x, e_n \rangle\, e_n. \tag{1}$$

We fix $k \geq 1$ and argue that

$$\begin{aligned} &\left\langle x - \sum_{n\leq k} \langle x, e_n \rangle\, e_n, \sum_{n\leq k} \langle x, e_n \rangle\, e_n \right\rangle \\ &= \left\langle x, \sum_{n\leq k} \langle x, e_n \rangle\, e_n \right\rangle - \left\langle \sum_{n\leq k} \langle x, e_n \rangle\, e_n, \sum_{n\leq k} \langle x, e_n \rangle\, e_m \right\rangle \\ &= \sum_{n\leq k} |\langle x, e_n \rangle|^2 - \sum_{n,m\leq k} \langle x, e_n \rangle \overline{\langle x, e_m \rangle} \delta_{nm} = 0. \end{aligned}$$

Hence, we can apply the Pythagorean theorem and deduce

$$\left\| x - \sum_{n \le k} \langle x, e_n \rangle e_n \right\|^2 + \left\| \sum_{n \le k} \langle x, e_n \rangle e_n \right\|^2 = \|x\|^2. \tag{2}$$

On the other hand, we have that

$$\left\| \sum_{k_1 \le n \le k_2} \langle x, e_n \rangle \, e_n \right\|^2 = \sum_{k_1 \le n \le k_2} \langle x, e_n \rangle \, \overline{\langle x, e_n \rangle} = \sum_{k_1 \le n \le k_2} | \langle x, e_n \rangle |^2 \tag{3}$$

for all $1 \le k_1 \le k_2$. These two facts jointly imply that

$$\sum_{n \ge 1} |\langle x, e_n \rangle|^2 < \infty$$

and, thus,

$$k \mapsto \sum_{n \le k} |\langle x, e_n \rangle|^2$$

is a Cauchy sequence. Factoring in again (3), it follows that

$$k \mapsto \sum_{n \le k} \langle x, e_n \rangle e_n$$

is a Cauchy sequence in X and, hence, convergent.

If we let

$$\lim_{k \to \infty} \sum_{n \le k} \langle x, e_n \rangle e_n = t,$$

then, for an arbitrary fixed $j \ge 1$, we derive

$$\begin{aligned} \langle x - t, e_j \rangle &= \lim_{k \to \infty} \langle x - \sum_{n \le k} \langle x, e_n \rangle e_n, e_j \rangle = \langle x, e_j \rangle - \lim_{k \to \infty} \sum_{n \le k} \langle x, e_n \rangle \delta_{nj} \\ &= \lim_{k \to \infty} \langle x, e_j \rangle - \langle x, e_j \rangle = 0. \end{aligned}$$

Using the completeness of $\{e_n\}_{n \ge 1}$, we obtain that $t = x$ and the proof of (1) is finished.

Secondly, we show the uniqueness of the representation of x as $\sum_{n \ge 1} x_n e_n$. By linearity of the limits in normed spaces, it is enough to show that

$$\sum_{n \ge 1} x_n e_n = 0 \implies x_n = 0, \quad (\forall)\, n \ge 1.$$

By definition, the left-hand side is equivalent to

$$\lim_{k \to \infty} \left\| \sum_{n \le k} x_n e_n \right\|^2 = 0.$$

However, one can obtain relatively easy that

$$\left\|\sum_{n\le k} x_n e_n\right\|^2 = \sum_{n\le k} |x_n|^2,$$

which implies

$$\lim_{k\to\infty} \sum_{n\le k} |x_n|^2 = \sum_{n\ge 1} |x_n|^2 = 0.$$

Thus, $x_n = 0$ for all $n \ge 1$ and the argument is completed. □

Next, we claim that both c_0 and l^p (with $1 \le p < \infty$) have $\{e_n\}_{n\ge 1}$ as a Schauder basis where

$$e_n = (0, \ldots, 0, 1, 0, \ldots)$$

and 1 is in the n-th position. We present the proof only for the space l^p.

Proposition 2. $\{e_n\}_{n\ge 1}$ defined above is a basis for l^p for any $1 \le p < \infty$.

Proof. First, we show that for any $x \in l^p$ one has

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

where

$$x = (x_1, x_2, \ldots, x_n, \ldots).$$

Indeed, for a fixed $m \ge 1$, we infer

$$x - \sum_{i=1}^{m} x_i e_i = (0, \ldots, 0, x_{m+1}, x_{m+2}, \ldots)$$

and, hence,

$$\left\|x - \sum_{i=1}^{m} x_i e_i\right\| = \left(\sum_{i=m+1}^{\infty} |x_i|^p\right)^{1/p}.$$

However, as $x \in l^p$, we derive

$$\left(\sum_{n\ge 1}^{\infty} |x_n|^p\right)^{1/p} = \|x\| < \infty$$

and, thus,

$$\lim_{m\to\infty} \sum_{i=m+1}^{\infty} |x_i|^p = 0.$$

This yields

$$\lim_{m\to\infty}\left\| x - \sum_{i=1}^{m} x_i e_i \right\| = 0$$

and the claim is proven.

To finish, we need to show the uniqueness of the representation $x = \sum_{n=1}^{\infty} x_n e_n$. For this purpose, like in the proof of the previous proposition, we invoke the linearity of limits and argue that it is enough to prove that

$$\sum_{n=1}^{\infty} a_n e_n = 0 \implies a_n = 0, \quad (\forall)\, n \geq 1.$$

By definition, the left-hand side is equivalent to

$$\lim_{k\to\infty}\left\| \sum_{n\leq k} a_n e_n \right\| = 0.$$

However, we can infer directly that

$$\left\| \sum_{n\leq k} a_n e_n \right\|^p = \sum_{n\leq k} |a_n|^p,$$

which implies

$$\sum_{n\geq 1} |a_n|^p = 0.$$

This obviously yields $a_n = 0$ for all $n \geq 1$ and the proof of the proposition is finished. □

To conclude this section, **we present a basis for** $C([0,1])$, which is not as natural and easy to find like the ones for the previously discussed Banach spaces. We start by listing the dyadic points in $[0,1]$ in the form of a sequence $\{t_n\}_{n\geq 1}$ with

$$t_1 = 0,\ t_2 = 1,\ t_3 = \frac{1}{2},\ t_4 = \frac{1}{4},\ t_5 = \frac{3}{4},\ t_6 = \frac{1}{8},\ t_7 = \frac{3}{8},\ t_8 = \frac{5}{8},\ \ldots$$

With the help of this sequence, we define the sequence of functions $\{f_n\}_{n\geq 1} \subset C([0,1])$ as follows:

$$\begin{aligned}
&f_1 \equiv 1, \quad f_2(t) = t,\ (\forall)\, t \in [0,1];\\
&n > 2, \quad f_n(t_j) = 0,\ (\forall) 1 \leq j < n; \quad f_n(t_n) = 1,\ \text{and} \qquad (4)\\
&f_n \text{ is linear between any consecutive points of the first } n \text{ dyadic points.}
\end{aligned}$$

The figure below contains the graphs of the first five terms of this sequence of functions.

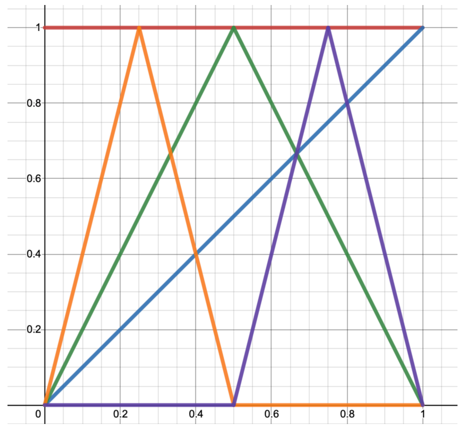

Figure 1: $\{f_n\}_{n=1}^5$

The main result we have here is:

Proposition 3. The sequence $\{f_n\}_{n\geq 1}$ represents a Schauder basis for $C([0,1])$.

Proof. In arguing for this result, we show first that if for some $g \in C([0,1])$ one has

$$g = \sum_{n\geq 1} a_n f_n, \tag{5}$$

then necessarily

$$a_1 = g(0), \qquad a_n = g(t_n) - \sum_{1\leq j\leq n-1} a_j f_j(t_n), \quad (\forall)\, n \geq 2. \tag{6}$$

This clearly proves the uniqueness of such representations for elements of $C([0,1])$. We recall that norm convergence in $C([0,1])$ is equivalent to uniform convergence, which implies pointwise convergence. Thus, from (5), we infer that

$$\lim_{k\to\infty} \sum_{n\leq k} a_n f_n(t_j) = g(t_j), \quad (\forall)\, j \geq 1.$$

By using (4) and choosing $k > j$, we deduce

$$\sum_{n\leq k} a_n f_n(t_j) = \sum_{n\leq j} a_n f_n(t_j),$$

which implies

$$\sum_{n\leq j} a_n f_n(t_j) = g(t_j), \quad (\forall)\, j \geq 1. \tag{7}$$

Therefore, by taking consecutive values of j in an increasing way, we obtain

$$\begin{aligned}
g(0) &= g(t_1) = a_1 f_1(t_1) = a_1, \\
g(t_j) &= \sum_{1\le n\le j-1} a_n f_n(t_j) \,+\, a_j f_j(t_j) = \sum_{1\le n\le j-1} a_n f_n(t_j) \,+\, a_j \\
&\Longrightarrow\; a_j = g(t_j) - \sum_{1\le n\le j-1} a_n f_n(t_j), \quad (\forall)\, j \ge 2
\end{aligned}$$

and this proves (6).

Now, all which is left to prove is that, for an arbitrary $g \in C([0,1])$, the representation (5) is indeed valid if $\{a_n\}_{n\ge 1}$ are given by (6). For this purpose, let us define

$$P_j = \sum_{n\le j} a_n f_n, \quad (\forall)\, j \ge 1,$$

and argue that

$$\lim_{j\to\infty} \|P_j - g\| = 0. \tag{8}$$

First, we rely on (4) and (6) to infer that

$$P_j(t_k) = \sum_{n\le j} a_n f_n(t_k) = \sum_{n\le k} a_n f_n(t_k) = g(t_k), \quad (\forall)\, k \le j. \tag{9}$$

Secondly, the first $N = 2^n + 1$ dyadic points are

$$0,\, 1,\, \frac{1}{2},\, \frac{1}{4},\, \frac{3}{4},\, \ldots,\, \frac{2^n - 1}{2^n},$$

and the distance between any two of their consecutive images (as graphed on the real line) is precisely $1/2^n$. Thus, if one considers the first $N \ge 2^n + 1$ dyadic points, the previous distance becomes at most equal to $1/2^n$.

Next, we fix $\epsilon > 0$ and, based on g being uniformly continuous on $[0,1]$ (as a continuous function on a compact set), deduce the existence of $\delta > 0$ such that

$$|g(x) - g(y)| < \epsilon, \quad (\forall)\, x, y \in [0,1],\ |x - y| < \delta.$$

Then, with the help of the Archimedean property, we choose $n_\delta \ge 1$ integer such that $1/2^{n_\delta} < \delta$ and take $N > 2^{n_\delta}$ to be an arbitrary integer. It follows from previous facts that if

$$\{t_1,\, t_2,\, \ldots, t_n\} = \{t_{j_1} < t_{j_2} < \ldots < t_{j_N}\},$$

then

$$|g(t_{j_i}) - g(t_{j_{i+1}})| < \epsilon, \quad (\forall) 1 \le i \le N - 1.$$

Let us consider now $x \in [t_{j_i}, t_{j_{i+1}}]$ and, since P_N is linear on $[t_{j_i}, t_{j_{i+1}}]$, we deduce from (9) that

$$|P_N(t_{j_i}) - P_N(x)| \le |P_N(t_{j_i}) - P_N(t_{j_{i+1}})| = |g(t_{j_i}) - g(t_{j_{i+1}})| < \epsilon.$$

On the other hand, since

$$|x - t_{j_i}| \leq |t_{j_i} - t_{j_{i+1}}| < \delta,$$

we derive from (3) that

$$|g(x) - g(t_{j_i})| < \epsilon.$$

Given that $P_N(t_{j_i}) = g(t_{j_i})$, the triangle inequality implies

$$|g(x) - P_N(x)| \leq |g(x) - g(t_{j_i})| + |P_N(t_{j_1}) - P_N(x)| < 2\epsilon.$$

As both the interval $[t_{j_i}, t_{j_{i+1}}]$ and $x \in [t_{j_i}, t_{j_{i+1}}]$ were chosen arbitrarily, we conclude that

$$\|P_N - g\| < 2\epsilon, \quad (\forall)\, N > 2^{n_\delta},$$

and, hence, the argument for (8) is finished. □

4 Another way of testing whether $\{e_n\}_{n\geq 1}$ forms a basis

By the definition of basis, we can see clearly that if $\{e_n\}_{n\geq 1}$ is a basis for X, we get $\overline{span\{e_n\}_{n\geq 1}} = X$. So, the next question would be to ask whether this is an if and only if statement. Unfortunately, even when we add the condition that each e_n is linearly independent, we cannot guarantee it forms a basis. One example of this is when we consider $X = C[0,1]$, and $\{e_n\}_{n\geq 1} = \{t^m\}_{m\geq 0}$ where $\{t^m\}_{m\geq 0}$ is the sequence of polynomials.

Therefore, the next natural question is what other conditions are needed for the sufficiency to be true. The following theorem would give us some idea.

Theorem 4.1. If $\{e_n\}_{n\geq 1} \subset X$ satisfying

$$\overline{span\{\{e_n\}_{n\geq 1}\}} = X,$$
$$e_n \neq 0, \forall n \geq 1,$$

then $\{e_n\}_{n\geq 1}$ is a basis for X if and only if $\exists K > 0, \forall n, p \in \mathbb{N}$ and scalars $\{a_i\}_{i\geq 1}$ such that

$$K||\sum_{i=1}^{n+p} a_i e_i|| \geq ||\sum_{i=1}^{n} a_i e_i||.$$

Proof. **First, we show the necessity**. In order to do so, we introduce a new norm $|||\cdot|||$, and use inverse mapping theorem on the two spaces $(X, |||\cdot|||)$ and $(X, ||\cdot||)$ to construct the K we are looking for.

Definition 4.1. $\forall x = \sum_{i=1}^{\infty} x_i e_i$ we define

$$|||x||| = \sup\left\{\left\|\sum_{i=1}^{n} x_i e_i\right\| : n \geq 1\right\}$$

To show it is a norm, we do the following three verification:

$$
\begin{aligned}
|||x||| = 0 &\iff \sup \left\{ \left\| \sum_{i=1}^{n} x_i e_i \right\| : n \geq 1 \right\} = 0 \\
&\iff \left\| \sum_{i=1}^{n} x_i e_i \right\| = 0, \forall n \geq 1 \iff x = 0; \\
|||ax||| &= \sup \left\{ a \left\| \sum_{i=1}^{n} x_i e_i \right\| : n \geq 1 \right\} \\
&= a \sup \left\{ \left\| \sum_{i=1}^{n} x_i e_i \right\| : n \geq 1 \right\} = a|||x|||; \\
|||x+y||| &= \sup \left\{ \left\| \sum_{i=1}^{n} (x_i + y_i) e_i \right\| : n \geq 1 \right\} \\
&\leq \sup \left\{ \left\| \sum_{i=1}^{n} x_i e_i \right\| + \left\| \sum_{i=1}^{n} y_i e_i \right\| : n \geq 1 \right\} \\
&\leq \sup \left\{ \left\| \sum_{i=1}^{n} x_i e_i \right\| : n \geq 1 \right\} + \sup \left\{ \left\| \sum_{i=1}^{n} y_i e_i \right\| : n \geq 1 \right\} \\
&= |||x||| + |||y|||.
\end{aligned}
$$

And this shows $|||\cdot|||$ is indeed a norm.

Then, **we want to show that X is complete given this new norm**, i.e. taking any Cauchy sequence $\{x^n\}_{n\geq 1} \subset X$ under the norm $|||\cdot|||$, want to show $\exists x \in X$ such that

$$\lim_{n\to\infty} |||x^n||| = |||x|||.$$

Since $\{x^n\}_{n\geq 1}$ is Cauchy, $\forall \epsilon > 0, \exists n_\epsilon$ such that

$$\forall n, m \geq n_\epsilon, |||x^n - x^m||| < \epsilon \text{ where } x^n, x^m \in X.$$

Notice that we can write $x^n = \sum_{k\geq 1} a_{n,k} e_k, x^m = \sum_{k\geq 1} a_{m,k} e_k$ for some scalars $\{a_{n,k}\}_{k\geq 1}, \{a_{m,k}\}_{k\geq 1}$. Therefore,

$$|||x^n - x^m||| = |||\sum_{k\geq 1} (a_{n,k} - a_{m,k}) e_k||| < \epsilon,$$

which implies

$$\forall t \geq 1, \left\| \sum_{k\leq t} (a_{n,k} - a_{m,k}) e_k \right\| < \epsilon. \tag{10}$$

Using 10 and triangle inequality, we get

$$\begin{aligned}||(a_{n,t}-a_{m,t})e_t|| &= \left\|\sum_{k\le t}(a_{n,k}-a_{m,k})e_k - \sum_{k\le t-1}(a_{n,k}-a_{m,k})e_k\right\| \\ &\le \left\|\sum_{k\le t}(a_{n,k}-a_{m,k})e_k\right\| + \left\|\sum_{k\le t-1}(a_{n,k}-a_{m,k})e_k\right\| \\ &< 2\epsilon.\end{aligned}$$

Therefore

$$|a_{n,t}-a_{m,t}| < \frac{2\epsilon}{||e_t||},$$

for all $n > m \ge n_\epsilon$ and for all t and thus, we get $\{a_{n,t}\}_{n\ge1}$ is Cauchy.

Let $b_t = \lim_{n\to\infty} a_{n,t}$, using 10 again, we get

$$\lim_{n\to\infty}\left\|\sum_{k\le t}(a_{n,k}-a_{m,k})e_k\right\| = \left\|\sum_{k\le t}(b_k-a_{m,k})e_k\right\| \le \epsilon, \tag{11}$$

for all $m \ge n_\epsilon$ and for all $t \ge 1$. Then, we want to show $\{\sum_{p\le n} b_p e_p\}_{n\ge1} \subset X$ is Cauchy, since once it is Cauchy, we would get $\sum_{p\ge1} b_p e_p \in X$, which will be the ideal candidate for x.

Notice that by triangle inequality $\forall n_1, n_2 \in \mathbb{N}$

$$\begin{aligned}\left\|\sum_{p\le n_1} b_p e_p - \sum_{p\le n_2} b_p e_p\right\| \le \left\|\sum_{p\le n_1}(b_p-a_{m,p})e_p\right\| + \left\|\sum_{p\le n_2}(b_p-a_{m,p})e_p\right\| \\ + \left\|\sum_{p\le n_1} a_{m,p}e_p - \sum_{p\le n_2} a_{m,p}e_p\right\|.\end{aligned}$$

By picking $m \ge n_\epsilon$, along with 11, we get

$$\left\|\sum_{p\le n_1}(b_p-a_{m,p})e_p\right\| + \left\|\sum_{p\le n_2}(b_p-a_{m,p})e_p\right\| \le 2\epsilon.$$

Also notice that $\sum_{k\ge1} a_{m,k}e_k$ is Cauchy, so we have $\forall\epsilon > 0, \exists N_\epsilon$, such that $\forall n_1, n_2 \ge N_\epsilon$

$$\left\|\sum_{p\le n_1} a_{m,p}e_p - \sum_{p\le n_2} a_{m,p}e_p\right\| < \epsilon.$$

These two facts jointly imply that $\forall\epsilon > 0, \exists N_\epsilon$, $\forall n_1, n_2 > N_\epsilon$

$$\left\|\sum_{p\le n_1} b_p e_p - \sum_{p\le n_2} b_p e_p\right\| \le 3\epsilon,$$

which implies

$$\sum_{p\geq 1} b_p e_p \in X.$$

Let $x = \sum_{p\geq 1} b_p e_p \in X$. By 11, we get

$$|||x - x^m||| = \sup_{n\geq 1} \left\| \sum_{p\leq n} (b_p - a_{m,p}) e_p \right\| \leq \epsilon,$$

for all $m \geq n_\epsilon, \forall t \geq 1$. Therefore, $x^m \to x$ under $|||\cdot|||$, hence X is complete under this norm, and $(X, |||\cdot|||)$ is Banach.

Define I to be the identity map from $(X, |||\cdot|||) \to (X, ||\cdot||)$. Trivially followed by how we define $|||\cdot|||$, I is a continuous mapping. Hence, by the **Inverse Mapping Theorem** I^{-1} is also continuous, and clearly, I^{-1} is also a linear map, so $|||I^{-1}|||$ is bounded. Which means $|||I^{-1}||| < \infty$.

Let $K = |||I^{-1}|||$, we have

$$|||x||| = |||I^{-1}x||| \leq |||I^{-1}||| \cdot ||x|| = K||x||,$$

for all $x \in X$. Therefore

$$\sup_{j\geq 1} \left\| \sum_{i\leq j} c_i e_i \right\| \leq K \left\| \sum_{i\geq 1} c_i e_i \right\|,$$

for all scalars $\{c_i\}_{i\geq 1}$.

In order to get $|| \sum_{i=1}^{n} a_i e_i || \leq K || \sum_{i\geq 1}^{n+p} a_i e_i ||$, for any scalars $\{a_i\}_{i\geq 1}, \forall n, p \in \mathbb{N}$, we define c_i as below

$$c_i = \begin{cases} a_i & i \leq n+p \\ 0 & i > n+p, \end{cases}$$

and, thus,

$$\sum_{i\leq j} c_i e_i = \sum_{i=1}^{\min\{j,n+p\}} a_i e_i,$$

$$\sum_{i\geq 1} c_i e_i = \sum_{i=1}^{n+p} a_i e_i.$$

Therefore, $|| \sum_{i=1}^{n} a_i e_i || \leq \sup_{j\geq 1} || \sum_{i=1}^{\min\{j,n+p\}} a_i e_i || \leq K || \sum_{i\geq 1}^{n+p} a_i e_i ||$.

Secondly, we show the **sufficiency**. In order to do so, we use the definition of a basis, which requires uniqueness of the scalars $\{x_n\}_{n\geq 1}$, and $x = \sum_{n=1}^{\infty} x_n e_n$.

Recall that **uniqueness** means for any scalars $\{b_i\}_{i\geq 1}, \{c_i\}_{i\geq 1}$,

$$\sum_{i\geq 1} b_i e_i = \sum_{i\geq 1} c_i e_i \implies b_i = c_i, \quad (\forall)\, i \geq 1$$

Nevertheless, $\sum_{i\geq 1} b_i e_i = \sum_{i\geq 1} c_i e_i$ gives us $\sum_{i\geq 1}(b_i - c_i)e_i = 0$. **It's now equivalent as showing for any scalars $\{a_i\}_{i\geq 1}$,**

$$\sum_{i\geq 1} a_i e_i = 0 \implies a_i = 0, \quad (\forall)\, i \geq 1.$$

In order to prove the statement above, we start off by defining $S_k = \sum_{i\leq k} a_i e_i$, with $\lim_{k\to\infty} S_k = 0$. Then, $\exists K$ such that for all $k_1 < k_2$, $||S_{k_1}|| \leq K||S_{k_2}||$. In particular, for all $l > 1$, we have $||S_1|| \leq K||S_l||$. And notice that

$$\lim_{l\to\infty} K||S_l|| = 0 \implies ||S_1|| \leq 0 \implies ||a_1 e_1|| = 0 \implies a_1 = 0.$$

Therefore, the base case is true. By weak induction, we suppose $\forall i \leq n, a_i = 0$, and want to show $a_{n+1} = 0$. In this case, for all $l > n+1$

$$||S_{n+1}|| = ||S_n + a_{n+1}e_{n+1}|| = ||a_{n+1}e_{n+1}|| \leq K||S_l||,$$

and by the exact argument as the base case, we get $a_{n+1} = 0$.

Hence, by induction we get $a_i = 0, \forall i \geq 1$. And the proof for uniqueness is completed.

We move on to proving $x = \sum_{n=1}^{\infty} x_n e_n$, and to do so we start off by defining

$$Y = \{x \in X : x = \sum_{i\geq 1} a_i e_i\},$$

and our goal is to show $Y = X$.

Notice that $\forall x \in span\{e_i\}_{i\geq 1}$ we can write

$$x = a_1 e_{i_1} + ... + a_k e_{i_k}, \text{ for some } k \geq 1$$

which is clearly in Y. Hence, $span\{e_i\}_{i\geq 1} \subset Y$. So if we can show Y is a closed linear space, we would get $X = \overline{span\{e_i\}_{i\geq 1}} \subseteq \overline{Y} = Y$.

To show Y is closed, we take any convergent sequence $\{x^n\}_{n\geq 1} \subset Y$, and want to show $x = \lim_{n\to\infty} x^n \in Y$.

Since $\{x^n\}_{n\geq 1} \subset Y$, we have $x^n = \sum_{k\geq 1} a_{n,k} e_k, x^m = \sum_{k\geq 1} a_{m,k} e_k$ for some scalars $\{a_{n,k}\}_{k\geq 1}, \{a_{m,k}\}_{k\geq 1}$. Then, $\exists K > 0$ such that

$$\left\| \sum_{k\leq t} (a_{n,k} - a_{m,k}) e_k \right\| \leq K \left\| \sum_{k\leq t+p} (a_{n,k} - a_{m,k}) e_k \right\|, \quad (\forall)\, t, p \in \mathbb{N}$$

And when taking $p \to \infty$, we have $\forall n, m \geq 1, t \in \mathbb{N}$

$$\left\| \sum_{k\leq t} (a_{n,k} - a_{m,k}) e_k \right\| \leq K \left\| \sum_{k\leq t+p} (a_{n,k} - a_{m,k}) e_k \right\| = ||x^n - x^m||. \tag{12}$$

By triangle inequality, we get

$$\begin{aligned} ||(a_{n,t} - a_{m,t})e_t|| &= \left\| \sum_{k \le t} (a_{n,k} - a_{m,k})e_k - \sum_{k \le t-1} (a_{n,k} - a_{m,k})e_k \right\| \\ &\le \left\| \sum_{k \le t} (a_{n,k} - a_{m,k})e_k \right\| + \left\| \sum_{k \le t-1} (a_{n,k} - a_{m,k})e_k \right\| \\ &\le 2K||x^n - x^m||, \end{aligned}$$

which implies

$$|a_{n,t} - a_{m,t}| \le \frac{2K||x^n - x^m||}{||e_t||}.$$

Using the fact that $\{x^n\}_{n \ge 1}$ is convergent, and hence, Cauchy, we have $\forall \epsilon > 0, \exists N$, $\forall n, m \ge N$

$$||x^n - x^m|| < \frac{||e_t||\epsilon}{2K}.$$

The above two facts jointly imply that

$$|a_{n,t} - a_{m,t}| < \epsilon.$$

and, thus $\{a_{n,t}\}_{n \ge 1}$ is Cauchy, hence, convergent. Then, for all $t \ge 1$, we define $b_t = \lim_{n \to \infty} a_{n,t}$, and want to show

$$x = \sum_{k \ge 1} b_k e_k,$$

By triangle inequality, we have

$$\left\| \sum_{k \le t} b_k e_k - x \right\| \le \left\| \sum_{k \le t} (b_k - a_{n,k})e_k \right\| + \left\| \sum_{k \le t} a_{n,k} e_k - x^n \right\| + ||x^n - x||.$$

Using 12, and take $n \to \infty$, we get $\forall t \ge 1$

$$\begin{aligned} \left\| \sum_{k \le t} (b_k - a_{n,k})e_k \right\| &= \lim_{m \to \infty} \left\| \sum_{k \le t} (a_{m,k} - a_{n,k})e_k \right\| \\ &\le \lim_{m \to \infty} K||x^m - x^n|| = K||x - x^n||. \end{aligned}$$

Therefore

$$\left\| \sum_{k \le t} b_k e_k - x \right\| \le (K+1)||x^n - x|| + \left\| \sum_{k \le t} a_{n,k} e_k - x^n \right\|.$$

Since $\lim_{n \to \infty} x^n = x$, we have

$$\forall \epsilon > 0, \exists N' > 0, \forall n > N', ||x^n - x|| < \frac{\epsilon}{2(K+1)}.$$

On the other hand, by definition of x^n, we have $\forall \epsilon > 0, \exists N'' > 0$

$$\forall t > N'', \left\| \sum_{k \le t} a_{n,k} e_k - x^n \right\| < \frac{\epsilon}{2}.$$

Hence $\forall t > N''$,

$$\left\| \sum_{k \le t} b_k e_k - x \right\| \le (K+1) \cdot \frac{\epsilon}{2(K+1)} + \frac{\epsilon}{2} = \epsilon,$$

which implies

$$x = \sum_{k \ge 1} b_k e_k.$$

Therefore, $x \in Y$, and hence, Y is a closed linear space. We have $X = Y$, which implies

$$\forall x \in X, x = \sum_{n=1}^{\infty} x_n e_n.$$

□

5 Applications of Theorem 4.1

In this section, we will use Theorem 4.1 to help us check whether $\{e_n\}_{n \ge 1}$ is indeed a basis for a Banach Space by two examples.

The first example is in the context of $L^p[0,1]$ space. We start by introducing the Haar System.

Definition 5.1. For any fixed $r \in \mathbb{N}$, we take a partition of $[0,1]$ into intervals of length $\frac{1}{2^r}$, denoted each sub-interval by $\{I_j^r : 1 \le j \le 2^r\}$. For $1 \le k \le 2^{r-1}$, we define $\chi_{2^{r-1}+k} = \chi_{I^r_{2k-1}} - \chi_{I^r_{2k}}$, and $\chi_1 = \chi_{(0,1)}$. The set $\{\chi_n\}_{n \ge 1}$ is called Haar system.

Notice that in Haar system

$\chi_1 = \chi_{(0,1)}$

$\chi_2 = \chi_{(0,\frac{1}{2})} - \chi_{(\frac{1}{2},1)}$

$\chi_3 = \chi_{(0,\frac{1}{4})} - \chi_{(\frac{1}{4},\frac{1}{2})}; \chi_4 = \chi_{(\frac{1}{2},\frac{3}{4})} - \chi_{(\frac{3}{4},1)}$

$\chi_5 = \chi_{(0,\frac{1}{8})} - \chi_{(\frac{1}{8},\frac{1}{4})}; \chi_6 = \chi_{(\frac{1}{4},\frac{3}{8})} - \chi_{(\frac{3}{8},\frac{1}{2})}; \chi_7 = \chi_{(\frac{1}{2},\frac{5}{8})} - \chi_{(\frac{5}{8},\frac{3}{4})}; \chi_8 = \chi_{(\frac{3}{4},\frac{7}{8})} - \chi_{(\frac{7}{8},1)}$

...

Remark.

$$\chi_{(0,1)} = \chi_1;$$
$$\chi_{(0,\frac{1}{2})} = \frac{\chi_1 + \chi_2}{2}, \chi_{(\frac{1}{2},1)} = \frac{\chi_1 - \chi_2}{2};$$
$$\chi_{(0,\frac{1}{4})} = \frac{\chi_{(0,\frac{1}{2})} + \chi_3}{2}, \chi_{(\frac{1}{4},\frac{1}{2})} = \frac{\chi_{(0,\frac{1}{2})} - \chi_3}{2},$$
$$\chi_{(\frac{1}{2},\frac{3}{4})} = \frac{\chi_{(\frac{1}{2},1)} + \chi_4}{2}, \chi_{(\frac{3}{4},1)} = \frac{\chi_{(\frac{1}{2},1)} - \chi_4}{2};$$
...

Corollary 5.0.1. Span of the Haar system covers $\chi_{I_j^r}, \forall r \geq 1, 1 \leq j \leq 2^r$.

Proof. We use induction on r to prove this Corollary.

Base case: When $r = 1$, we have

$$\chi_{(0,\frac{1}{2})} = \frac{\chi_1 + \chi_2}{2}, \chi_{(\frac{1}{2},1)} = \frac{\chi_1 - \chi_2}{2};$$

Suppose it is true for all $r \leq t$, want to show it's true for $r = t + 1$. Notice that $\forall 1 \leq j \leq 2^{t+1}$, if we look at the interval $(\frac{j}{2^{t+1}}, \frac{j+1}{2^{t+1}})$, we know at least one of $j, j+1$ is even.

If j is even, let $p = \frac{j}{2}$, we know

$$\frac{j}{2^{t+1}} = \frac{p}{2^t}, \frac{j+2}{2^{t+1}} = \frac{p+1}{2^t} \implies \left(\frac{j}{2^{t+1}}, \frac{j+1}{2^{t+1}}\right) \subsetneq \left(\frac{p}{2^t}, \frac{p+1}{2^t}\right)$$

By weak induction, we know

$$\chi_{(\frac{p}{2^t},\frac{p+1}{2^t})} \in span\{\chi_n\}.$$

On the other hand, there exists $\alpha \in N$ such that

$$\chi_\alpha = \chi_{(\frac{j}{2^{t+1}},\frac{j+1}{2^{t+1}})} - \chi_{(\frac{j+1}{2^{t+1}},\frac{j+2}{2^{t+1}})} \in span\{\chi_n\}_{n \geq 1}.$$

Therefore

$$\chi_{(\frac{j}{2^{t+1}},\frac{j+1}{2^{t+1}})} = \frac{\chi_{(\frac{p}{2^t},\frac{p+1}{2^t})} + \chi_\alpha}{2} \in span\{\chi_n\}_{n \geq 1}.$$

A similar argument would work for $j + 1$ is even.

Hence, by weak induction, $span\{\chi_n\}_{n \geq 1}$ covers $\chi_{I_j^r}, \forall r \geq 1, 1 \leq j \leq 2^r$. □

Corollary 5.0.2. Dyadic points are dense in $[0, 1]$.

Proof. For any $x \in [0, 1]$, we can write it in binary form $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$. Then, each partial sum $\sum_{i=1}^{n} \frac{x_i}{2^i}$ is a dyadic point. Therefore, x can be approximated by a sequence of dyadic point. Hence, the dyadic points are dense. □

Proposition 4. $\{\chi_n\}_{n\geq 1}$ is a Basis for $L^p[0,1]$, for any fixed $p, 1 \leq p < \infty$.

Proof. **In order to use Theorem 4.1, we first show that** $\overline{span\{\{\chi_n\}_{n\geq 1}\}} = L^p[0,1]$. We let A be the set of dyadic points.

Remark. A has Lebesgue measure 0.

Remark. The end points $I_j^r, \forall r \geq 1, 1 \leq j \leq 2^r$ are in A.

By *Corollary 5.0.1 and Corollary 5.0.2*, we have $\overline{A} = [0,1]$, and hence,

$$\forall I \subseteq [0,1], I \in \overline{span\{\chi_n\}_{n\geq 1}}.$$

Therefore, $\overline{span\{\chi_n\}_{n\geq 1}} = L^p([0,1])$.

Secondly, we want to show $\chi_n \not\equiv 0, \forall n \geq 1$**.** But this is follows trivially from $\chi_n = \chi_{I_j^r} - \chi_{I_{j+1}^r}$, for some $r \geq 1, j \geq 1$, and the fact that $\mu(I_j^r) = \mu(I_{j+1}^r) = \frac{1}{2^r} > 0$.

Lastly, we are left to show $\exists K > 0, \forall n, p \in \mathbb{N}, \exists \{a_i\}_{i\geq 1}$

$$K||\sum_{i=1}^{n+p} a_i e_i|| \geq ||\sum_{i=1}^{n} a_i e_i||.$$

In order to do so, let's prove something stronger, that is $K||\sum_{i=1}^{n+1} a_i\chi_i|| \geq ||\sum_{i=1}^{n} a_i\chi_i||$, which will trivially give us what we want.

We define

$$f = \sum_{i=1}^{n} a_i\chi_i, g = \sum_{i=1}^{n+1} a_i\chi_i = f + a_{n+1}\chi_{n+1}.$$

and want to show $\exists K > 0$ such that $K||g|| \geq ||f||$.

Remark. For all $x \in [0,1]$, if we have $\chi_{n+1}(x) = 0$, then $g(x) = f(x)$, i.e., for all $x \in [0,1] \setminus \text{supp}\chi_{n+1}, g(x) = f(x)$.

Let $\text{supp}\chi_{n+1}$ be I_{n+1}. We have

$$||f||^p = \int_{[0,1]} |f|^p = \int_{[0,1]\setminus I_{n+1}} |f|^p + \int_{I_{n+1}} |f|^p$$
$$||g||^p = \int_{[0,1]} |g|^p = \int_{[0,1]\setminus I_{n+1}} |f|^p + \int_{I_{n+1}} |g|^p$$

Hence, it is enough to show $\int_{I_{n+1}} |g|^p \geq \int_{I_{n+1}} |f|^p$. Since I_{n+1} is the interval where $\chi_{n+1} \neq 0$, we know there exists two disjoint sub-intervals I'_{n+1}, I''_{n+1} where $I'_{n+1} \cup I''_{n+1} = I_{n+1}$, and $\mu(I'_{n+1}) = \mu(I''_{n+1}) = \frac{\mu(I_{n+1})}{2}$, with $\chi_{n+1} = 1$ on I'_{n+1} and $\chi_{n+1} = -1$ on I''_{n+1}, which implies

$$g = f + a_{n+1} \text{ on } I'_{n+1}$$
$$g = f - a_{n+1} \text{ on } I''_{n+1}$$

and, hence,

$$\int_{I_{n+1}} |g|^p = \int_{I'_{n+1}} |g|^p + \int_{I''_{n+1}} |g|^p$$
$$= \int_{I'_{n+1}} |f + a_{n+1}|^p + \int_{I''_{n+1}} |f - a_{n+1}|^p$$

Remark. $\forall j \in \mathbb{N}, j < n+1$, we have either $I_j \cap I_{n+1} = \phi$ or $I_{n+1} \subset I_j$ with either $I_{n+1} \subseteq I'_j$ or $I_{n+1} \subseteq I''_j$.

In all cases, we would get χ_j is constant on I_{n+1}. Hence, we can let $f = \alpha$, for some constant α on the interval I_{n+1}, and this gives us

$$\int_{I_{n+1}} |f|^p = \int_{I_{n+1}} |\alpha|^p = |\alpha|^p \mu(I_{n+1})$$

and

$$\begin{aligned}\int_{I_{n+1}} |g|^p &= \int_{I'_{n+1}} |f + a_{n+1}|^p + \int_{I''_{n+1}} |f - a_{n+1}|^p \\ &= |\alpha + a_{n+1}|^p \mu(I'_{n+1}) + |\alpha - a_{n+1}|^p \mu(I''_{n+1}) \\ &= |\alpha + a_{n+1}|^p \frac{\mu(I_{n+1})}{2} + |\alpha - a_{n+1}|^p \frac{\mu(I_{n+1})}{2} \\ &= (|\alpha + a_{n+1}| + |\alpha - a_{n+1}|) \frac{\mu(I_{n+1})}{2}.\end{aligned}$$

Therefore, in order to show $\int_{I_{n+1}} |g|^p \geq \int_{I_{n+1}} |f|^p$, we are left to show

$$|\alpha + a_{n+1}|^p + |\alpha - a_{n+1}|^p \geq 2|\alpha|^p = 2|\frac{(\alpha + a_{n+1}) + (\alpha - a_{n+1})}{2}|^p.$$

But this follows easily from the convexity of $T(x) = |x|^p$ when $p \geq 1$ (which can be proven by mean value theorem). Therefore,

$$||f||^p \leq ||g||^p \implies ||f|| = \left[\int_{[0,1]} |f|^p\right]^{\frac{1}{p}} \leq \left[\int_{[0,1]} |g|^p\right]^{\frac{1}{p}} = ||g||$$

Hence, $K = 1$ works and that $\forall n, p \in \mathbb{N}$,

$$||\sum_{i=1}^{n} a_i \chi_i|| \leq ||\sum_{i=1}^{n+1} a_i \chi_i|| \leq ... \leq ||\sum_{i=1}^{n+p-1} a_i \chi_i|| \leq ||\sum_{i=1}^{n+p} a_i \chi_i||.$$

By Theorem 4.1, we have Haar system forms a basis for $L^p[0,1]$. □

Nevertheless, it is not always the case that using Theorem 4.1 can help us make the proof easier. One example of Theorem 4.1 does not work as we expected is the basis we found in *section 3*, the dyadic points forms a basis for $C[0,1]$. Recall in section 3, we proved this is a basis using the definition of the basis. Here, we take a look at how we can prove it by applying Theorem 4.1.

Proof. Let $\{f_n\}_{n\geq 1}, P_n$ be defined the same as the previous section, in order to use Theorem 4.1, **we need to prove three things, the first is**

$$\overline{span\{f_n\}_{n\geq 1}} = C[0,1].$$

Secondly,

$$f_n \not\equiv 0, \forall n \geq 1.$$

Thirdly, $\exists K > 0$**,** $\forall n, p \in \mathbb{N}, \exists \{a_i\}_{i\geq 1}$**, such that**

$$K||\sum_{i=1}^{n+p} a_i f_i|| \geq ||\sum_{i=1}^{n} a_i f_i||.$$

As for $\overline{span\{f_n\}_{n\geq 1}} = C[0,1]$, it can be shown using the same argument when we show $\lim_{k\to\infty} ||P_k - g|| = 0$ as $P_k \in \overline{span\{f_n\}_{n\geq 1}}$. (Which is most of the proofs needed for it to become a basis using definition, as the part of showing uniqueness is trivial.) And now, we still need to check the non-zeroness, and the existence of K.

The non-zeroness part follows directly from the definition of f_n, as $f_n(t_n) = 1 \neq 0, \forall n \geq 1$, which implies $f_n \not\equiv 0$.

To show $K||\sum_{i=1}^{n+p} a_i f_i|| \geq ||\sum_{i=1}^{n} a_i f_i||$, it is equivalent as showing $K||P_{n+p}|| \geq ||P_n||$. Notice that for all $j \leq n$

$$P_{n+p}(t_j) = \sum_{i=1}^{n} a_i f_i(t_j) + \sum_{i=n+1}^{n+p} a_i f_i(t_j) = P_n(t_j) + \sum_{i=n+1}^{n+p} a_i f_i(t_j).$$

However, for all $i > j, f_i(t_j) = 0$, which implies

$$\sum_{i=n+1}^{n+p} a_i f_i(t_j) = 0,$$

and, hence,

$$P_{n+p}(t_j) = P_n(t_j), \qquad (\forall)\ j \leq n. \tag{13}$$

Then, we **claim** for any fixed $n \geq 1$, $\max\{|P_n(t_j)| : j \geq 1\} = |P_n(t_i)|$ for some $i \leq n$.

<u>Proof:</u> Suppose on the contrary, for all $i \leq n, \max\{|P_n(t_j)| : j \geq 1\} > |P_n(t_i)|$, we have

$$\exists i' > n, \text{ such that } |P_n(t_{i'})| > |P_n(t_i)|.$$

Nevertheless, since $i' > n \geq i$ we have there exists two consecutive points (in the sense of the real line) t_a, t_b where $a, b \leq n$ such that

$$t_{i'} \in (t_a, t_b) \text{ but } t_i \notin (t_a, t_b).$$

By linearity of P_n on (t_a, t_b) we have

$$\min\{P_n(t_a), P_n(t_b)\} \leq P_n(t_{i'}) \leq \max\{P_n(t_a), P_n(t_b)\},$$

which implies
$$|P_n(t_{i'})| \leq \max\{|P_n(t_a)|, |P_n(t_b)|\}.$$
Therefore,
$$|P_n(t_{i'})| \leq |P_n(t_i)| \text{ for some } i \leq n,$$
which gives us an contradiction. Therefore
$$||P_n|| = \max\{|P_n(t_j)| : j \geq 1\} = |P_n(t_i)| \text{ for some } i \leq n$$

■

Using this claim, along with 13, we have
$$||P_{n+p}|| \geq |P_{n+p}(t_j)| = |P_n(t_j)| = ||P_n||.$$
Hence, when $K = 1$, we have $K||P_{n+p}|| \geq ||P_n||$, for all $n, p \in \mathbb{N}$. □

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