Geodesic Curves on the Lorentz Lie Group Equipped with a Positive Non-invariant Metric

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Abstract

The Euler-Arnold equation describes the geodesics on a Lie group on which a Riemmanian metric is induced, through left group translation, by a given inner product defined on the corresponding Lie algebra. In particular, Arnold used this equation to show that ideal fluids can be understood as geodesics on the infinite-dimensional Lie bi-algebra of incompressible vector fields.

In this paper, the same approach is applied to the computation of the geodesic curves on the Lorentz Lie group of relativistic transformations. The corresponding Lorentz Lie algebra is indeed a simple example of finite-dimensional Lie bi-algebra, so that this example becomes a "toy model" of the case of ideal fluids described above.

In order to obtain a positive metric on the Lorentz group which is expected to give an integrable system, we define a positive, non-invariant inner product on the Lorentz Lie bi-algebra. Then, we derive the geodesic equations of motion in a convenient system of coordinates which takes advantage of the Lie bi-algebra structure.

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1 Literature Review

"Nothing takes place in the world whose meaning is not that of some maximum or minimum."
- Leonhard Euler

1.1 Lie Groups and Lie Algebras

Lie groups, named after Sophus Lie, are the mathematical objects which describe continuous symmetries; that is, continuous transformations which preserve a prescribed property of a system. A typical example is three-dimensional rotational symmetry as the class of continuous isometries of a sphere in three dimensions.

More formally, a Lie group G consists of a group that is also a differentiable manifold [1]. By the group structure, the Lie group contains an identity element e, and every element g in the Lie group has a multiplicative inverse g^{-1} which composes with it to give the identity: $gg^{-1} = I = g^{-1}g$. These conditions ensure that the identity map is a symmetry transformation, and that every symmetry transformation can be reversed to bring the object back to its original configuration. As a manifold, the Lie group is locally diffeomorphic to Euclidean space R^n, which in particular implies continuity of G in the sense that elements of G can get "arbitrarily closer" to each other (as opposed to elements of a discrete space). Finally, it must be enforced that the operations of composition μ (also called multiplication) and inversion i be smooth (i.e. infinitely differantiable) in order for them to respect the diffeomorphic character of the Lie manifold:

$$\mu : G \to G \times G; \quad \mu(x; y) = xy$$

and

$$i : G \to G; \quad I(x) = x^{-1}$$

are smooth maps.

The previous paragraph presents the complete set of conditions characterizing Lie groups, which have proven to be extensively useful to the study continuous symmetries and their applications; not only in mathematics but also different areas of theoretical physics, where continuous symmetries lead to conserved quantities of motion by Noether's Theorem. To illustrate this idea, we formally present the $SO(3)$ group of three-dimensional rotations about the origin as our first example of a Lie group:

Example 1. $SO(3) = \{Q \in GL(3; R) \mid QQ^T = I$ and $\det Q = 1\}$.

The condition $QQ^T = I$ means that Q is orthogonal and hence preserves distances, while $\det Q = 1$ excludes the rotations, which have $\det Q = -1$. Note that $\det Q \neq 0$ ensures that all elements in $SO(3)$ will have an inverse in the group. The check that matrix multiplication and matrix inversion in SO(3) are smooth operations is straightforward and omitted here.

In the same context of rotations just considered, Sophus Lie realized that a general tranformation is generated by an infinitesimal one by virtue of the continuity of the group. That is, any finite rotation $R(\theta) \in SO(3)$ around a given axis can be divided in N portions by considering N successive applications of $R(\theta/N)$ (that is, $(R(\theta/N))^N$). Letting $N \to \infty$, $R(\theta/N)$ becomes an infinitesimal rotation that can be written as $R(d\theta) = I + A + O(d\theta^2)$, where A is of order $d\theta$. This statement is generally true for all Lie groups. With this infinitesimal expansion, the orthogonality condition $RR^T = I$ becomes

$$I = (I + A)(I + A)^T = I + A^T + A + O(d\theta^2) \tag{1}$$

which gives the antisymmetric condition $A^T = -A$ on A. Then, A can be expanded in terms of a basis J_1, J_2, J_3 of three-dimensional antisymmetric matrices as:

$$A = \theta_1 J_1 + \theta_2 J_2 + \theta_3 J_3$$

where

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

Back to Eq. 1, the relation between A and $R(\theta)$ is the following:

$$R(\theta) = (R(\theta/N))^N = (I + A)^N = \left(I + \frac{\theta_1 J_1 + \theta_2 J_2 + \theta_3 J_3}{N}\right)^N$$

and taking the limit gives the exponential formula for $R(\theta)$ in terms of the generators J_i (which explains their name):

$$R(\theta) = exp\left(\frac{\theta_1 J_1 + \theta_2 J_2 + \theta_3 J_3}{N}\right)$$

It is straightforward to check that $\theta_i J_i$ corresponds to a rotation by θ_i around the $i'th$ axis, so that the coefficients θ_i specify all the information about the rotation that we initially represented by θ. From this, it is clear that the non-commutability of $SO(3)$ is encoded in the commutation relation of the J_i generators. These facts are true for any Lie group, and lead to the concept of the Lie algebra associated with a Lie group, which we define next.

For the purposes of this paper, the Lie algebra $\mathfrak{g}$ associated to a Lie group G is the tangent space of G at the identity. More formally, a Lie algebra is defined as the vector space of the generators of G, together with a Lie bracket operation: an alternating bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}; (U, V) \to [U, V]$, called the Lie bracket or commutator map, which represents anti-commutation and satisfies the Jacobi identity:

$$[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0. \tag{3}$$

The properties of $\mathfrak{g}$ that will be useful to us are contained in the commutation relations dictated by the Lie bracket. For matrix groups such as those considered in this paper, the commutator of two arbitrary matrices U, V assumes the simple form $[U, V] = UV - VU$. In general, since the image of the Lie bracket is also in $\mathfrak{g}$, it follows that we can expand $[U_i, U_j] = \sum_k \epsilon_{ijk} U_k$ for any pair $U_i, U_j \in \mathfrak{g}$; the coefficients ϵ_{ijk} are called the structure constants, and contain the information that we are interested in. Therefore, for a first example of a Lie algebra, let us return to the case of rotations and compute the commutation relations of the generators J_i.

Example 2. $\mathfrak{so}(3)$: Lie algebra of SO(3)

$$[J_1, J_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = J_3$$

$$\text{and similarly } [J_2, J_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = J_1 \text{ and } [J_3, J_1] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = J_2. \tag{4}$$

The remaining cases follow by anticommutation of the Lie bracket. Thus, we see that the structure constants for $\mathfrak{so}(3)$ are simply the components of the antisymmetric Levi-Civita tensor $_{ijk}$ in three dimensions, so that the Lie bracket in this case corresponds to a vector cross-product. This algebra, for instance, is the basis for the theory of angular momentum in quantum mechanics.

1.2 The Lorentz Lie Group and Algebra

It is the aim of this paper to find and solve the equations for geodesics on the Lorentz Lie group. Consequently, this section presents the Lorentz Lie group and its associated Lie algebra.

To begin with, the Lorentz group $O(1,3)$ is the group formed by Lorentz transformations, which relate physical frames of reference that move with respect to each other at a given constant velocity. These transformations, named after the physicist Hendrik Lorentz, encapsulate the symmetry of the four-dimensional Minkowski spacetime, which serves as stage for the theory of special relativity. In other words, the Lorentz transformations are precisely those that preserve the Minkowski distance $c^2(t_2 \quad t_1)^2 + (x_2 \quad x_1)^2 + (y_2 \quad y_1)^2 + (z_2 \quad z_1)^2$ between any two events $E_1 = (t_1, x_1, y_1, z_1), E_2 = (t_2, x_2, y_2, z_2),$ in spacetime. It is clear that the spatial rotations in $SO(3)$ satisfy this condition when embedded in four dimensions (acting as the identity on the time component, so that the J_i are extended with first rows and columns filled with zeros). There is another type of Lorentz transformations, known as Lorentz boosts, which are generated by the following matrices:

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Of course, any composition of spatial rotations and Lorentz boosts also represents some element of the Lorentz Lie algebra, this is, some operator that generates an element of $O(1,3)$ through exponentiation. In fact, the six J_i and K_i are a basis for the Lorentz algebra $\mathfrak{o}(1,3)$, and these are usually taken as canonical generators.

Nonetheless, it is often more convenient to work with the representation of the Lorentz group in terms of two-by-two matrices, obtained by identifying Minkowski four-vectors of the form (t, x, y, z) with the matrices $\begin{pmatrix} t+z & x \quad iy \\ x+iy & t \quad z \end{pmatrix}$, so that the determinant of the matrix gives the Minkowski norm of the corresponding vector. In this representation, the six Lorentz generators take up the following form:

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; J_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$K_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};\; K_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\text{ and } K_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

However, for the purpose of computation in this work, we choose the alternative basis $\{X_i\}$ given by the following six elements:

$$X_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix};\; X_2 = \begin{pmatrix} 0 & 2i \\ 0 & 0 \end{pmatrix};\; X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$X_4 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix};\; X_5 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\;\text{ and } X_6 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{5}$$

Note first that X_4, X_5, X_6 are just the J_1, J_2, J_3 multiplied by $-i$, so that they are proportional to the three Pauli matrices; while the X_1, X_2, X_3 are an altogether different a choice of basis from the infinitesimal boosts K_i. Here we often emphasize this distinction between the first three and the second three basis elements by referring to the former as the upper-triangular basis elements, and the latter as the diagonal basis elements. In this way, the new basis makes the bi-algebra structure of $\mathfrak{o}(1,3)$ explicit (see [2]). In short, a Lie bi-algebra is a Lie algebra enhanced by a co-bracket operation, which also satisfies antilinearity, the co-Jacobi identity similar to 3, and a compatibility condition with the Lie bracket. We will only need to take advantage of the compatibility condition of the bi-algebra structure, in order to simplify our computations; this condition will be introduced and used later on, in Section 2.2.

1.3 Lie Geometry and the Euler-Arnold Equation

In this section, we consider the geometry of Lie groups on which a notion of distance is defined through a Riemannian metric g, which assigns an inner product $\langle u, v\rangle_{g(x)}$ to every pair of tangent vectors $u, v \in T_xG$, the tangent space of G (as a manifold) at x, for all points $x \in G$. In particular, the metric defines a non-negative length $|v|_{g(x)} = \sqrt{\langle v, v\rangle_{g(x)}} \in \mathbb{R}^+$ for every tangent vector v. Besides, in order to be a well-defined metric, this inner product is required to be symmetric, positive-definite, and smooth as a function of x. With respect to the interaction between the metric and the group structure of G, we say that the metric g is left-invariant if it is invariant under left translation or multiplication:

$$\langle u, v\rangle_{g(x)} = \langle (dL_y)_x v, (dL_y)_x w\rangle_{g(yx)}$$

for all $x, y \in G$ and all $u, v \in T_xG$. Here, dL_y is the differential of the map that left-multiplies by y.

We now introduce the concept of geodesic curve. With respect to a pair of endpoints $a, b \in G$, and given a prescribed metric, a geodesic from a to b is any smooth curve $\gamma : [a, b] \to G$ which minimizes the length

$$|\gamma| := \int_a^b |\gamma'(t)|_{g(\gamma(t))} dt.$$

In order to remove the degeneracy in this definition induced by reparameterizations of γ, it is more convenient to consider the equivalent condition of energy minimization instead of the length [3], with the energy similarly given by the integral of the square of the speed:

$$E(\gamma) := \frac{1}{2}\int_a^b |\gamma'(t)|^2_{g(\gamma(t))}\,dt. \tag{6}$$

This condition is equivalent to the previous one under the assumption that the speed $|\gamma'(t)|$ remains constant throughout the curve. Therefore, we define a geodesic flow or geodesic curve to a curve that minimizes the value of E for fixed points a, b. Then, in the context of Lagrangian mechanics, geodesic flows describe the trajectories of particles that move within G in the absence of external mechanical forces.

A key observation about the geometry of Lie groups is that the tangent space at any point x is actually determined by the tangent space at the identity (i.e. the Lie algebra), through the left translation by x provided by the group structure. This means that we can choose an inner product on the Lie algebra, and use the product it induces on the tangent bundle TG as left-invariant a Riemannian metric on G, in a way that will be made precise in Eq. 7 below. Then, it turns out that our initial choice of inner product on the Lie algebra actually determines the geodesic flows on G through the induced inner product, such that the geodesics can a priori be computed directly from a differential equation involving the Lie algebra structure. Such procedure was pioneered by Vladimir Arnold in his 1966 paper [4], where he showed that many important physical equations of motions can be viewed as geodesic equations on a Lie group equipped with a left-invariant metric of the type just considered. The general form of the geodesic equation in this context is known as the Euler-Arnold equation, and two celebrated examples of particular instances are the Euler equations of motion of a rigid body and the Euler equations of fluid dynamics of an incompressible fluid with no viscosity.

Now, for $v \in T_xG$ a tangent vector at point $x \in G$, the corresponding element $dx^{-1}(v)$ in the Lie algebra $\mathfrak{g}$ is given by the pullback of v by x. Conversely, any element $U \in \mathfrak{g}$ can be transported to a tangent vector at x through the pushforward $dx(v)$. Therefore, given an inner product $\langle u, v\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ we can promote it to a metric given by

$$\langle dL_x(u), dL_x(v)\rangle_{g(x)} = \langle u, v\rangle_{\mathfrak{g}} \tag{7}$$

for all $x \in G$. This metric is automatically left-invariant, by virtue of its definition as an extension by left-translation of the inner product at the identity.

There is another bilinear form that is relevant to the geodesic flows in the Euler-Arnold picture. This is the partial adjoint $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of the Lie bracket (with respect to the chosen inner product):

$$\langle [U, V], W\rangle = \langle B(W, V), U\rangle \tag{8}$$

Indeed, the form B describes the geodesic flow through the Euler-Arnold equation, which we present next.

Theorem 1 (Euler-Arnold equation). Let $\gamma : R \to G$ be a geodesic flow on G for the left-invariant metric g defined above, and let $V(t) = \gamma^{-1}(t)\gamma'(t) \in \mathfrak{g}$ be its intrinsic velocity vector. Then V is given by the equation

$$\frac{d}{dt}v(t) = B(v(t), v(t)). \tag{9}$$

Theorem 1 represents the central result employed in this paper. We conclude this subsection with its proof as given by Tao [3] (except that Tao uses the convention corresponding to a right-invariant metric, which flips all terms involving γ^{-1} in the theorem and throughout the proof).

Proof of Theorem 1

Consider a variation $\gamma(s,t)$ of the original curve $\gamma(t) = \gamma(0,t)$ and find the first variation of the energy

$$\partial_s \frac{1}{2} \int_a^b \langle \partial_t \gamma(t), \partial_t \gamma(t) \rangle_{g(\gamma(s,t))} dt.$$

which can be rewritten by Eq. 8 as:

$$\partial_s \frac{1}{2} \int_a^b \langle \gamma^{-1}\gamma_t, \gamma^{-1}\gamma_t \rangle dt = \int_a^b \langle \partial_s(\gamma^{-1}\gamma_t), \gamma^{-1}\gamma_t \rangle dt$$

where the last equality makes use of the linearity and symmetry of the product to bring the variational derivative into one of the product factors. Substituting in $V(t)$ we obtain

$$\int_a^b \langle \partial_s(\gamma^{-1}\gamma_t), V \rangle dt.$$

Expanding the variational derivative gives

$$\partial_s(\gamma^{-1}\gamma_t) = -\gamma^{-2}\gamma_s\gamma_t + \gamma^{-1}\gamma_{ts} = \gamma^{-1}\gamma_{ts} - \gamma^{-1}\gamma_s\gamma^{-1}\gamma_t$$

and similarly

$$\partial_t(\gamma^{-1}\gamma_s) = \gamma^{-1}\gamma_{st} - \gamma^{-1}\gamma_t\gamma^{-1}\gamma_s$$

Since the partial derivatives commute ($\gamma_{st} = \gamma_{ts}$), we have found

$$\partial_s(\gamma^{-1}\gamma_t) = \partial_t(\gamma^{-1}\gamma_s) + [\gamma^{-1}\gamma_s, V]$$

We substitute this back into the energy variation, and after integration by parts we get

$$\int_a^b \langle \gamma^{-1}\gamma_s, B(V,V) \rangle - \langle \gamma^{-1}\gamma_s, \partial_t V \rangle dt.$$

We want the first variation to vanish for an arbitrary choice of γ_s, which by the expression just found occurs exactly when the Euler-Arnold equation $\partial_t V(t) = B(V(t), V(t))$ holds.

1.4 Matrix Inner Products

The previous section made apparent that the choice of inner product on a Lie algebra shapes the Arnold geodesic flows on the corresponding Lie group. In this case, we are interested on a matrix Lie group. Thus, we use this section to introduce two common matrix inner products, including the one chosen in this paper to define our metric, before beginning the actual geodesic calculations in the next section.

To begin with, there is a bi-invariant but non-positive matrix inner product [2], given by

$$(U,V) = [Tr(UV)]. \tag{10}$$

This is a complex-valued invariant product, and in fact its real and imaginary parts represent two independent invariant products. This fact will become important when we consider the constants of motion or conserved quantities related to our geodesic trajectories. The reason for this is that invariant products generate conserved quantities, as we will prove belows; thus, the two parts of this complex invariant product will furnish two independent constants of motion.

Besides, the imaginary part of this product associates the following scalars to each pair of elements in our Lie bi-algebra basis $\{X_i\}$ from 5:

$$(X_i, X_j) = 0 = (X^i, X^j), (X_i, X^j) = \delta_i^j$$

These simple relations are a consequence of the fact that the Lie bi-algebra $o(1,3)$ can be decomposed into two isotropic subalgebras, spanned by respectively by $\{X_1, X_2, X_3\}$ and $\{X_4, X_5, X_6\}$ see (Example 2.1 of [2]).

However, the geodesic flows induced by an invariant product itself are largely uninteresting, because this property of the product makes the metric too simple (in a sense, the invariance of the metric is trivial in this case), and ultimately leads to trivial geodesic equations. For this reason, to define our metric we use the Frobenius matrix inner product, which is non-invariant but positive:

$$\langle U, V\rangle = Re[Tr(U^\dagger V)]. \tag{11}$$

As the next section will show, the Frobenius products of our basis elements are highly symmetric, which leads to conservation laws that ultimately can be used to analytically solve the system formed by the geodesic equations.

2 Inner-product and Commutation Relations in the Lorentz Lie Algebra

This section contains the computations of all inner products and commutators of the basis elements $\{X_i\}$. These will be used in the next section in order to solve the Euler-Arnold equation 9, which involves both products and commutators through the definition of B in 8, in the coordinate system fixed by this basis.

2.1 Frobenius Inner-product Elements

There is a total of $6 \times 6 = 36$ inner-product elements for our six-element basis. Those products involving only X_4, X_5, X_6, the diagonal basis elements, are most easily found by remembering their proportionality relation with the Pauli matrices, $X_{i+3} = -\frac{i}{2}\sigma_i$, where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we can use their hermiticity and trace properties, $\sigma_i^\dagger = \sigma_i$ and $Tr[\sigma_i \sigma_j] = \delta_{ij} I$, to find:

$$\langle X_{3+i}, X_{3+j}\rangle = Re\left[Tr\left(X_{3+i}^\dagger X_{3+j}\right)\right] = Re\left[-\left(\frac{i}{2}\right)^2 Tr\left(\sigma_i^\dagger \sigma_j\right)\right] = \frac{1}{4}Re\left[Tr\left(\delta_{ij} I\right)\right] = \frac{1}{2}\delta_{ij}$$

where $i, j = 1, 2, 3$. This expression thus covers nine of the thirty-six products. The remaining twenty-seven, by the symmetry of the product, can be found through the following $(27 - 3)/2 + 3 = 15$

different computations:

$$\langle X_1, X_1\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix}\right)\right] = Re\left(0^2 + (-2)^2\right) = 4;$$

similarly

$$\langle X_1, X_2\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix}\right)\right] = Re[-4i] = 0;$$

$$\langle X_1, X_3\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\right] = Re[0] = 0;$$

$$\langle X_2, X_2\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2i & 0\end{pmatrix}\begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix}\right)\right] = Re[4] = 4;$$

$$\langle X_2, X_3\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\right] = Re[0] = 0;$$

$$\langle X_3, X_3\rangle = Re\left[Tr\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)\right] = Re[2] = 2;$$

$$\langle X_1, X_4\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re\left(\frac{i}{2}(-2) + 0\right) = 0;$$

$$\langle X_1, X_5\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re[-1] = -1;$$

$$\langle X_1, X_6\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}\frac{i}{2} & 0\\ 0 & -\frac{i}{2}\end{pmatrix}\right)\right] = Re[0+0] = 0;$$

$$\langle X_2, X_4\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2i & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re\left(\frac{i}{2}(-2i) + 0\right) = 1;$$

$$\langle X_2, X_5\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ -2i & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re[-i] = 0;$$

$$\langle X_2, X_6\rangle = Re\left[Tr\left(\begin{pmatrix}0 & 0\\ 2i & 0\end{pmatrix}\begin{pmatrix}\frac{i}{2} & 0\\ 0 & \frac{i}{2}\end{pmatrix}\right)\right] = Re[0+0] = 0,$$

$$\langle X_3, X_4\rangle = Re\left[Tr\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re[0+0] = 0,$$

$$\langle X_3, X_5\rangle = Re\left[Tr\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\right)\right] = Re[0+0] = 0,$$

and lastly

$$\langle X_3, X_6\rangle = Re\left[Tr\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{i}{2} & 0\\ 0 & \frac{i}{2}\end{pmatrix}\right)\right] = Re\left[\frac{i}{2}(\ 1) + \frac{i}{2}(1)\right] = 0.$$

Finally, we collect all of the computed product elements, in our basis, into a 6-by-6 matrix given by $_{ij} = \langle X_i, X_j\rangle$:

$$= \begin{pmatrix} 4 & 0 & 0 & 0 & 1 & 0\\ 0 & 4 & 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0\\ 1 & 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\end{pmatrix}, \tag{12}$$

which is seen to be symmetric and checked to be positive by finding its eigenvalues in the usual way (in particular, they are: $4.265564437074637, 0.23443556292536255, 4.265564437074637, 0.23443556292536255$, $2,$ and 0.5).

2.2 Commutation Relations

This subsection provides the commutator relations in our basis. For a start, we use again the proportionality of our diagonal basis elements with the $_i$, since the Pauli matrices have well-known commutation relations. Therefore, we directly find (by bilinearity of the Lie bracket):

$$[X_{i+3}, X_{j+3}] = \left[\frac{i}{2}\ _i, \frac{i}{2}\ _j\right] = \left(\frac{i}{2}\right)^2 [\ _i, \ _j] = \frac{1}{4}(2i)\ _{ijk}\ _k = \ _{ijk}\left(\frac{i}{2}\ _k\right) = \ _{ijk}(X_{k+3}).$$

Then, we compute directly the commutators involving only the upper-triangular basis matrices:

$$[X_1, X_2] = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} = 0$$

$$[X_1, X_3] = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix} = 2\begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix} = 2X_2$$

and letting $2i \to -2$ in this computation we find similarly

$$[X_2, X_3] = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & +2 \\ 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = 2X_1$$

In a Lie bi-algebra, the commutators within both subalgebras actually determine the mixed commutators (in this case, commutators of a upper-triangular basis element with a diagonal one), through the compatibility condition. Letting a, b, c range over the indices $1, 2, 3$ for the upper-triangular basis and x, y, z over $4, 5, 6$ for the diagonal basis, the structure constants Γ^c_{ab} and Γ^z_{xy} within each subalgebra are given by

$$[X_a, X_b] = \sum_c \Gamma^c_{ab} X_c, \quad [X_x, X_y] = \sum_z \Gamma^z_{xy} X_z.$$

In this notation, the mixed commutators follow from the compatibility condition

$$[X_a, X_x] = \sum_{b=1,2,3} \Gamma^a_{x-3,b} X_b - \sum_{y=4,5,6} \Gamma^x_{a+3,y} X_y.$$

For convenience, we gather here the subalgebra commutators previously computed

$$\begin{aligned} &[X_1, X_2] = 0, && [X_1, X_3] = 2X_2, && [X_2, X_3] = 2X_1 \\ &[X_4, X_5] = X_6, && [X_4, X_6] = -X_5, && [X_5, X_6] = X_4. \end{aligned} \tag{13}$$

Combining these with the compatibility condition just introduced, we compute the mixed commutators explicitly:

$$\begin{aligned} &[X_1, X_4] = -2X_6, && [X_2, X_4] = -X_3, && [X_3, X_4] = X_2 + 2X_4 \\ &[X_1, X_5] = X_3, && [X_2, X_5] = -2X_6, && [X_3, X_5] = -X_1 + 2X_5 \\ &[X_1, X_6] = -X_6, && [X_2, X_6] = X_1, && [X_3, X_6] = 0. \end{aligned} \tag{14}$$

Every other commutation relation either follows directly from 13 and 14, or equals zero trivially by $[X_i, X_i] = 0$. As usual, we now define the structure constants Γ^k_{ij} for the Lorentz Lie algebra to be the coefficients appearing on the right-hand side of the commutation relations, according to $[X_i, X_i] = \Gamma^k_{ij} X_k$, where i, j, k range over $1, 2, 3, 4, 5, 6$. These structure constants enter the Euler-Arnold equation for geodesics through the adjoint B of the Lie bracket, when the latter is expressed in the coordinates corresponding to our $\{X_i\}$ basis.

3 Coordinate Form of the Euler-Arnold Equation

3.1 Derivation of the Coordinate Form of the Equation

This section presents the computations necessary to express, and ultimately solve, the Euler-Arnold equation (Eq. 9) for geodesics, in the system of coordinates fixed by our choice of basis $\{X_i\}$. For a start, we note that the Euler-Arnold equation is cast in coordinate form by projecting it into each basis element applying the Frobenius inner product:

$$\langle X_i, \frac{dv}{dt} \rangle = \langle X_i, B(v, v) \rangle$$

Expand the element v in our basis as $v = \sum_j v^j X_j$. Then, we can rewrite the equation above in terms of these coordinates, making use of the product elements 12 and Lie structure constants 13 and 14 and the definition 8 of B:

$$\langle X_i, \frac{dv^j}{dt} X_j \rangle = \langle [X_i, v], v \rangle = \langle [X_i, v^j X_j], v^l X_l \rangle = \langle v^j \Gamma^k_{ij} X_k, v^l X_l \rangle$$

$$\rightarrow \quad \Lambda_{ij} \frac{dv^j}{dt} = \Gamma^k_{ij} \Lambda_{kl} v^j v^l. \tag{15}$$

where linearity of both the Lie bracket and the product has been used repeatedly to take the coefficients v_j out. Also, here and moving forward, the Einstein summation convention is used for convenience. In this coordinate form, there is one such geodesic equation for each of the six components i. In general, the right hand side of the equation has a complicated quadratic form that makes the equations analytically unsolvable. In those cases, a numerical or approximate solution is often simple and sheds light on some aspects of the geodesic flow. In our particular case, however, the different symmetries of the inner product and Lie bracket (inherited by the tensors Γ^k_{ij} and Λ_{ij}) conspire to simplify greatly the form of this equations, by making most of the terms on the right hand side either equal zero or cancel mutually with other terms.

Example 1. Geodesic Equations for the Rigid Body.
In order to illustrate the use of the use of Eq. 15, we apply it in the case of the rigid body to derive Euler's equations of motions. In that case, $\Gamma^k_{ij} = \epsilon_{ijk}$, and the inner-product takes the form $\Lambda = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ for some constants I_i, in the basis formed by the usual rotations generators J_i.
The geodesic equations are then computed from these according to Eq. 15:

$$\begin{aligned}
\Lambda_{1j} \frac{dv^j}{dt} &= I_1 \frac{dv^1}{dt} = \Gamma^k_{1j} \Lambda_{kl} v^j v^l = \Gamma^3_{12} \Lambda_{33} v^2 v^3 + \Gamma^2_{13} \Lambda_{22} v^3 v^2 = (I_3 - I_2) v^2 v^3 \\
\Lambda_{2j} \frac{dv^j}{dt} &= I_2 \frac{dv^2}{dt} = \Gamma^k_{2j} \Lambda_{kl} v^j v^l = \Gamma^1_{23} \Lambda_{11} v^3 v^1 + \Gamma^3_{21} \Lambda_{33} v^1 v^3 = (I_1 - I_3) v^1 v^3 \\
\Lambda_{3j} \frac{dv^j}{dt} &= I_3 \frac{dv^3}{dt} = \Gamma^k_{3j} \Lambda_{kl} v^j v^l = \Gamma^2_{31} \Lambda_{22} v^1 v^2 + \Gamma^1_{32} \Lambda_{11} v^2 v^1 = (I_2 - I_1) v^1 v^2
\end{aligned}$$

which are Euler's equations of motions for a rotating rigid body with moments of inertia I_1, I_2, I_3 and angular velocity $\vec{\omega} = (v^1, v^2, v^3)$. When I_1, I_2, and I_3 are all different, which corresponds to a completely anisotropic rigid body, the solutions to these equations are given in terms of the Jacobi elliptic functions: sn(t), cn(t), dn(t) [5]. The Jacobi elliptic functions are doubly periodic and characterized by the property that the derivative of any one of the three is proportional to the product of the other two. In this sense, the Jacobi elliptic functions are a triplet generalization of the pair formed by the trigonometric functions sine and cosine.

Now, if the rigid body exhibits a spatial symmetry consisting of rotations within some plane, there exists a particular choice of basis that exploits this symmetry to simplify the equations of motions. For instance, we can choose axis 1 and 2 to lie in the symmetry plane, to render $I_1 = I_2$. In this case, the third equation above simplifies to $\frac{dv^3}{dt} = 0$, leading to the conservation of v^3. Then, the other two equations are effectively linear: the derivative of one of the two first components is linear in the other, which leads to solutions in terms of sine and cosine functions. In addition, finally, it is straightforward to note that a full rotational symmetry requires all moments I_i to be equal, which in turn leads to all components of $v(t)$ being conserved for a geodesic flow. In the next section, we will discover that our choice of inner product on the Lorentz Lie algebra is symmetric enough to render effectively linear equations (as in the second case above) with trigonometric solutions. We note that this kind of partially symmetric systems are more treatable than the non-symmetric counterpart, and more interesting that the trivial ones with complete symmetry.

3.2 Preliminary Analysis of Symmetries and Expected Conserved Quantities

In this subsection, we introduce the potential quantities that we expect to be conserved by the geodesic flow, making our system integrable.

In the Poisson-bracket formalism of Hamiltonian mechanics, the Poisson bracket $\{f, g\}$ is an antisymmetric bilinear function analogous to the Lie commutator $[\,,\,]$. More concretely, suppose that f, g are functions of the coordinates v^i on a Lie algebra with some basis (that is, f, g are functionals on the Lie algebra), and that Γ_k^{ij} are the usual structure constants related to the Lie commutator. Then, the Poisson bracket is related to the commutator through

$$\{f, g\} = v^k \Gamma_k^{ij} \frac{df}{dv^i} \frac{dg}{dv^j}. \tag{16}$$

Now, a quantity of motion is conserved if and only if its Poisson bracket with the Hamiltonian H are zero, and H itself is determined by the defined inner product according to $H = \langle V, V \rangle = \langle v^i X_i, v^j X_j \rangle = {}_{ij} v^i v^j$. For a start, the Hamiltonian H has zero bracket with itself (in the same way any quantity has zero commutator with itself), and is therefore always conserved. In our particular case, we know the product coefficients ${}_{ij}$ are given by the definition of in Eq. 12. This leads to the first conserved quantity, the Hamiltonian $H = 4v^1v^1 + 2v^1v^5 + 4v^2v^2 - 2v^2v^4 + 2v^3v^3 + \frac{1}{2}v^4v^4 + \frac{1}{2}v^5v^5 + \frac{1}{2}v^6v^6$.

We expect two more conserved quantities that are quadratic in the v^i, coming from the real and imaginary parts of the invariant inner product in 10. These are $Im[\Omega] = Im[(V, V)] = v^1v^4 + v^2v^5 + v^3v^6$, and $Re[\Omega] = Re[(V, V)] = -2v^1v^5 + 2v^2v^4 + 2v^3v^3 - \frac{1}{2}v^4v^4 - \frac{1}{2}v^5v^5 - \frac{1}{2}v^6v^6$. Of course, we must include here the proof that invariant products are conserved; in fact, we show that a product is invariant if and only if it has zero Poisson brackets with *all* coordinate components (which in particular implies our claim) as follows:

$$\begin{aligned} 0 &= \{\Omega, v^k\} = \Omega_{ij} v^i \{v^j, v^k\} \\ &= \Omega_{ij} v^i v^l \Gamma_l^{jk} = \frac{1}{2} v^i v^l [\Omega_{ij} \Gamma_l^{jk} + \Omega_{lj} \Gamma_i^{jk}] \end{aligned}$$

where we have used 16, and then exploited the symmetry consisting on exchanging the indices i and l. Since $v^i v^l$ are arbitrary coordinates, the above is equivalent to

$$\Omega_{ij} \Gamma_l^{jk} + \Omega_{lj} \Gamma_i^{jk} = 0,$$

this is,

$$([v^i X_i, v^j X_j], v^k X_k) + (v^j X_j, [v^i X_i, v^k X_k]) = 0$$

which is the statement of invariance, concluding the proof. Therefore, we anticipate the existence of three independent quadratic constants of motion associated with our dynamical system of geodesic flows. In fact, a straightforward calculation at this stage (which we omit in this text) shows that

H has zero Poisson brackets with itself and both parts of Ω have zero Poisson brackets with each v^i component, which provides a first check for our computations in Section 2.

The linear conserved quantities are trickier to identify at this stage, but simply foreseeing the existence of a number of them allow us to predict the integrability of the system. To understand this, let us use an analogy with the rigid body studied as an example in the previous subsection. In that case, there are three degrees of freedom $\{v^1, v^2, v^3\}$ and at least one conserved quantities: the system's Hamiltonian $H = I_1 v^1 v^1 + I_2^2 v^2 + I_3 v^3 v^3$. Depending on the presence or lack of symmetries, we might have from zero to two additional independent constants of motion such as v^3, v^1; because once two components are conserved, the third one is conserved automatically as a combination of H and the other two, so it is not independent. So the total number of (independent) conserved quantities is between one and three, and this number determines the simplicity of the dynamical system and its solutions, which are (respectively): Jacobi elliptic functions, sinusoidal functions, and trivial constants. In comparison, $O(1,3)$ has six degrees of freedom, and at least three constants of motion by the discussion above. In addition, the Lorentz Lie algebra contains a completely-symmetric rigid body as the subalgebra determined by $\{X_4, X_5, X_6\}$. This is expected to give rise to three conserved generators (for a total of six conserved quantities for six equations), though the particular form of this generators is unknown at this stage (in particular, they might represent pure rotations or combinations of rotations and boosts). Another hint for the existence of linear conserved quantities is the discrete symmetry present in the Lie algebra under the exchanges $X_1 \leftrightarrow \; X_2, X_4 \leftrightarrow X_5, X_3 \to X_3, X_6 \to \; X_6$, which can be seen to conserve all the inner products and commutators in Section 2, and might be a manifestation of an underlying rotational symmetry within some plane. Even if only two linear conserved quantities were present (five total conserved quantities for six equations), an analytical solution would be available in terms of Jacobi elliptic functions, and all three conservation laws would lead to a simpler solution in terms of exponential or even trigonometric functions (by analogy with the first and second cases of the rigid body). Indeed, after the explicit calculation in the next subsection, we will find that our equations include three conservation laws for three independent linear combinations of the v^i, which as expected makes our system integrable and straightforward to solve.

3.3 Geodesic Equations on the Lorentz Group

Here we present the derivation of the geodesic equations (in their coordinate form) for the Lorentz group with our choice of basis and Frobenius inner product. Compared to the analogous computation for the rigid body, carried out in Example 1, the right-hand side of our equations will in principle contain many more terms, as indices in this case range over six values rather than three. However, many of the terms will be zero because both the structure constants Γ and the entries of vanish for several index combinations. We take advantage of this fact to simplify the presentation of our computation, by following the following equation formatting: for each value of index i (which denotes the equation number), we look at the structure constants of the form Γ^k_{ij}, also for one value of j at a time. Then, given i and j, there are only at most two values of k for which Γ^k_{ij} does not vanish. For each of these values, Γ^k_{ij} couples to two terms proportional to one or two entries of the form ${}_{kl}$ for some l; after listing the non-vanishing values of Γ^k_{ij} for different k, we include a right-pointing arrow that introduces the non-vanishing contribution of the form $\Gamma^k_{ij}\;{}_{kl} v^j v^l$ to the right hand side of our geodesic equation number i. Then, we sum all these contributions over j to obtain the complete right-hand side of the equation. We repeat this process for each i.

i = 1:

The terms on the right-hand side can be found as follows:

$$\Gamma^{k}_{11} = 0$$
$$\Gamma^{k}_{12} = 0$$
$$\Gamma^{k}_{13} = 2 \quad {}_{k1} \qquad ! \quad \Gamma^{1}_{13} \quad {}_{11}v^3v^1 + \quad {}_{15}v^3v^5 \quad = 2 \quad 4v^3v^1 + 1v^3v^5$$
$$\Gamma^{k}_{14} = \quad 2 \quad {}_{k6} \qquad ! \quad \Gamma^{6}_{14} \quad {}_{66}v^4v^6 \quad = \quad 2 \quad 1{=}2v^4v^6$$
$$\Gamma^{k}_{15} = \quad {}_{k3} \qquad ! \quad \Gamma^{3}_{15} \quad {}_{33}v^5v^3 \quad = 1 \quad 2v^5v^3$$
$$\Gamma^{k}_{16} = \quad {}_{k2} \qquad ! \quad \Gamma^{2}_{16} \quad {}_{22}v^6v^2 + \quad {}_{24}v^6v^4 \quad = \quad 1 \quad 4v^6v^2 \quad 1v^6v^4 \quad :$$

Regarding the left-hand side:

$$ {}_{1j}\frac{dv^j}{dt} = \quad {}_{11}\frac{dv^1}{dt} + \quad {}_{15}\frac{dv^5}{dt} = 4\frac{dv^1}{dt} + \frac{dv^5}{dt}:$$

And setting them equal to each other:

$$4\frac{dv^1}{dt} + \frac{dv^5}{dt} = 2 \quad 4v^3v^1 + 1v^3v^5 \quad 2 \quad 1{=}2v^4v^6 \quad + 1 \quad 2v^5v^3 \quad 1 \quad 4v^6v^2 \quad 1v^6v^4 \quad = 8v^3v^1 + 4v^3v^5 \quad 4v^6v^2$$

$$! \quad 4\frac{dv^1}{dt} + \frac{dv^5}{dt} = 8v^3v^1 + 4v^3v^5 \quad 4v^6v^2 \tag{G.1}$$

This is the first of our geodesic equations. Each of the other five cases require a computation similar to this one, as presented below.

i = 2:

Right-hand side:

$$\Gamma^{k}_{21} = 0$$
$$\Gamma^{k}_{22} = 0$$
$$\Gamma^{k}_{23} = 2 \quad {}_{k2} \qquad ! \quad \Gamma^{2}_{23} \quad {}_{22}v^3v^2 + \quad {}_{24}v^3v^4 \quad = 2 \quad 4v^3v^2 \quad 1v^3v^4$$
$$\Gamma^{k}_{24} = \quad {}_{k3} \qquad ! \quad \Gamma^{3}_{24} \quad {}_{33}v^4v^3 \quad = \quad 1 \quad 2v^4v^3$$
$$\Gamma^{k}_{25} = \quad 2 \quad {}_{k6} \qquad ! \quad \Gamma^{6}_{25} \quad {}_{66}v^5v^6 \quad = \quad 2 \quad 1{=}2v^5v^6$$
$$\Gamma^{k}_{26} = \quad {}_{k1} \qquad ! \quad \Gamma^{1}_{26} \quad {}_{11}v^6v^1 + \quad {}_{15}v^6v^5 \quad = 1 \quad 4v^6v^5 + 1v^6v^5 \quad :$$

Left-hand side:

$$ {}_{2j}\frac{dv^j}{dt} = \quad {}_{22}\frac{dv^2}{dt} + \quad {}_{24}\frac{dv^4}{dt} = 4\frac{dv^2}{dt} \quad \frac{dv^4}{dt}:$$

And putting them together:

$$4\frac{dv^2}{dt} \quad \frac{dv^4}{dt} = 2 \quad 4v^3v^2 \quad 1v^3v^4 \quad 1 \quad 2v^4v^3 \quad 2 \quad 1{=}2v^5v^6 \quad + 1 \quad 4v^6v^5 + 1v^6v^5 \quad = 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1$$

$$! \ 4\frac{dv^2}{dt} \quad \frac{dv^4}{dt} = 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1 \tag{G.2}$$

i = 3:

Right-hand side:

$$\Gamma^k_{31} = \quad 2 \ {}_{k1} \qquad ! \ \Gamma^1_{31} \quad {}_{11}v^1v^1 + \ {}_{15}v^1v^5 \ = \quad 2 \ \ 4v^1v^1 + 1v^1v^5$$

$$\Gamma^k_{32} = \quad 2 \ {}_{k2} \qquad ! \ \Gamma^2_{32} \quad {}_{22}v^2v^2 + \ {}_{24}v^2v^4 \ = \quad 2 \ \ 4v^2v^2 \quad 1v^2v^4$$

$$\Gamma^k_{33} = 0$$

$$\Gamma^k_{34} = \ {}_{k2} + 2 \ {}_{k4} \qquad ! \ \Gamma^2_{34} \quad {}_{22}v^4v^2 + \ {}_{24}v^4v^4 \ + \Gamma^4_{34} \quad {}_{42}v^4v^2 + \ {}_{44}v^4v^4 \ = 1 \ \ 4v^4v^2 \quad 1v^4v^4 \ + 2 \quad 1v^4v^2 + \frac{1}{2}v^4v^4$$

$$\Gamma^k_{35} = \quad {}_{k1} + 2 \ {}_{k5} \qquad ! \ \Gamma^1_{35} \quad {}_{11}v^5v^1 + \ {}_{15}v^5v^5 \ + \Gamma^5_{35} \quad {}_{51}v^5v^1 + \ {}_{55}v^5v^5 \ = \quad 1 \ \ 4v^5v^1 + 1v^5v^5 \ + 2 \ \ 1v^5v^1 + \frac{1}{2}v^5v^5$$

$$\Gamma^k_{36} = 0:$$

Left-hand side:

$$ {}_{3j}\frac{dv^j}{dt} = \quad {}_{33}\frac{dv^3}{dt} = 2\frac{dv^3}{dt}:$$

Therefore:

$$2\frac{dv^3}{dt} = \quad 2 \ \ 4v^1v^1 + 1v^1v^5 \qquad 2 \ \ 4v^2v^2 \quad 1v^2v^4 \ + 1 \ \ 4v^4v^2 \quad 1v^4v^4 \ + 2 \quad 1v^4v^2 + \frac{1}{2}v^4v^4$$
$$1 \ \ 4v^5v^1 + 1v^5v^5 \ + 2 \ \ 1v^5v^1 + \frac{1}{2}v^5v^5$$

And after cancelling out terms:

$$2\frac{dv^3}{dt} = \quad 8v^1v^1 \quad 4v^1v^5 \quad 8v^2v^2 + 4v^2v^4 \tag{G.3}$$

i = 4:

Right-hand side:

$$\Gamma^k_{41} = 2 \ {}_{k6} \qquad ! \ \Gamma^6_{41} \quad {}_{66}v^1v^6 \ = 2 \quad \frac{1}{2}v^1v^6$$

$$\Gamma^k_{42} = \ {}_{k3} \qquad ! \ \Gamma^3_{42} \quad {}_{33}v^2v^3 \ = +1 \ \ 2v^4v^3$$

$$\Gamma^k_{43} = \quad {}_{k2} \quad 2 \ {}_{k2} \qquad ! \ \Gamma^2_{43} \quad {}_{22}v^3v^2 + \ {}_{24}v^3v^4 \ + \Gamma^4_{43} \quad {}_{42}v^3v^2 + \ {}_{44}v^3v^4 \ = \quad 1 \ \ 4v^3v^2 \quad 1v^3v^4 \quad 2 \quad 1v^3v^2 + \frac{1}{2}v^3v^4$$

$$\Gamma^k_{44} = 0$$

$$\Gamma^k_{45} = \ {}_{k6} \qquad ! \ \Gamma^6_{45} \quad {}_{66}v^5v^6 \ = 1 \ \ 1{=}2v^5v^6$$

$$\Gamma^k_{46} = \quad {}_{k5} \qquad ! \ \Gamma^5_{46} \quad {}_{51}v^6v^1 + \ {}_{55}v^6v^5 \ = \quad 1 \quad 1v^6v^1 + \frac{1}{2}v^6v^5 \quad :$$

Left-hand side:

$$ {}_{4j}\frac{dv^j}{dt} = \quad {}_{42}\frac{dv^2}{dt} + \ {}_{44}\frac{dv^4}{dt} = \quad \frac{dv^2}{dt} + \frac{1}{2}\frac{dv^4}{dt}:$$

Then we find:

$$\frac{dv^2}{dt}+\frac{1}{2}\frac{dv^4}{dt}=2\quad \frac{1}{2}v^1v^6\quad +1\ \ 2v^4v^3\qquad 1\ \ 4v^3v^2\quad 1v^3v^4$$
$$2\qquad 1v^3v^2+\frac{1}{2}v^3v^4\quad +1\ \ 1{=}2v^5v^6\qquad 1\quad 1v^6v^1+\frac{1}{2}v^6v^5$$

And since all the right-hand side terms cancel out:

$$! \qquad \frac{dv^2}{dt}+\frac{1}{2}\frac{dv^4}{dt}=0 \tag{G.4}$$

i = 5:

Right-hand side:

$$\Gamma^k_{51}=\qquad {}_{k3}\qquad\qquad ! \ \ \Gamma^3_{51}\quad {}_{33}v^1v^3\ =\ \ 1\ \ 2v^1v^3$$
$$\Gamma^k_{52}=2\ {}_{k6}\qquad\qquad ! \ \ \Gamma^6_{52}\quad {}_{66}v^2v^6\ =+2\ \ \frac{1}{2}v^2v^6$$
$$\Gamma^k_{53}=\ {}_{k1}\quad 2\ {}_{k5}\quad ! \ \Gamma^1_{53}\quad {}_{11}v^3v^1+\ {}_{15}v^3v^5\ +\Gamma^5_{53}\quad {}_{51}v^3v^1+\ {}_{55}v^3v^5\ =+1\ \ 4v^3v^1+1v^3v^5\qquad 2\quad +1v^3v^1+\frac{1}{2}v^3v^5$$
$$\Gamma^k_{54}=\qquad {}_{k6}\qquad\qquad ! \ \ \Gamma^6_{54}\quad {}_{66}v^4v^6\ =\ \ 1\ \ 1{=}2v^4v^6$$
$$\Gamma^k_{55}=0$$
$$\Gamma^k_{56}=\ {}_{k4}\qquad\qquad ! \ \ \Gamma^4_{56}\quad {}_{42}v^6v^2+\ {}_{44}v^6v^4\ =+1\qquad 1v^6v^1+\frac{1}{2}v^6v^4\quad :$$

Left-hand side:

$${}_{5j}\frac{dv^j}{dt}=\ {}_{51}\frac{dv^1}{dt}+\ {}_{55}\frac{dv^5}{dt}=\frac{dv^1}{dt}+\frac{1}{2}\frac{dv^5}{dt}:$$

Putting them together:

$$\frac{dv^1}{dt}+\frac{1}{2}\frac{dv^5}{dt}=\quad 1\ \ 2v^1v^3\quad +2\quad \frac{1}{2}v^2v^6\quad +1\ \ 4v^3v^1+1v^3v^5$$
$$2\quad +1v^3v^1+\frac{1}{2}v^3v^5\qquad 1\ \ 1{=}2v^4v^6\quad +1\qquad 1v^6v^1+\frac{1}{2}v^6v^4$$

And hence (as in the the previous case):

$$! \quad \frac{dv^1}{dt}+\frac{1}{2}\frac{dv^5}{dt}=0 \tag{G.5}$$

i = 6:

Right-hand side:

$$
\begin{aligned}
&\Gamma^k_{61} = \;\; {}_{k2} && ! \;\; \Gamma^2_{61} \;\; {}_{22}v^1v^2 + \;\; {}_{24}v^1v^4 \;\; = +1 \;\; 4v^1v^2 \quad 1v^1v^4 \\
&\Gamma^k_{62} = \quad {}_{k1} && ! \;\; \Gamma^1_{62} \;\; {}_{11}v^2v^1 + \;\; {}_{15}v^2v^5 \;\; = \quad 1 \;\; 4v^2v^1 + 1v^2v^5 \\
&\Gamma^k_{63} = 0 && \\
&\Gamma^k_{64} = \;\; {}_{k5} && ! \;\; \Gamma^5_{64} \;\; {}_{51}v^4v^1 + \;\; {}_{55}v^4v^5 \;\; = +1 \quad 1v^4v^1 + \frac{1}{2}v^4v^5 \\
&\Gamma^k_{65} = \quad {}_{k4} && ! \;\; \Gamma^4_{65} \;\; {}_{42}v^5v^2 + \;\; {}_{44}v^5v^4 \;\; = \quad 1 \quad\;\; 1v^5v^2 + \frac{1}{2}v^5v^4 \\
&\Gamma^k_{66} = 0:
\end{aligned}
$$

Left-hand side:

$$
{}_{6j}\frac{dv^j}{dt} = \;\; {}_{66}\frac{dv^6}{dt} = \frac{1}{2}\frac{dv^6}{dt}:
$$

And then:

$$
\frac{1}{2}\frac{dv^6}{dt} = +1 \;\; 4v^1v^2 \quad 1v^1v^4 \quad\; 1 \;\; 4v^2v^1 + 1v^2v^5 \;\; +1 \;\; 1v^4v^1 + \frac{1}{2}v^4v^5 \quad\; 1 \quad\; 1v^5v^2 + \frac{1}{2}v^5v^4
$$

$$
! \;\; \frac{1}{2}\frac{dv^6}{dt} = 0 \tag{G.6}
$$

Thus we find the resulting geodesic equations for the Lorentz Lie group.

Geodesic Equations, Eqs. G.1-G.6:

$$
\begin{aligned}
4\frac{dv^1}{dt} + \frac{dv^5}{dt} &= 8v^3v^1 + 4v^3v^5 \quad 4v^6v^2 \\
4\frac{dv^2}{dt} \quad \frac{dv^4}{dt} &= 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1 \\
2\frac{dv^3}{dt} &= \quad 8v^1v^1 \quad 4v^1v^5 \quad 8v^2v^2 + 4v^2v^4 \\
\frac{dv^2}{dt} + \frac{1}{2}\frac{dv^4}{dt} &= 0 \\
\frac{dv^1}{dt} + \frac{1}{2}\frac{dv^5}{dt} &= 0 \\
\frac{1}{2}\frac{dv^6}{dt} &= 0
\end{aligned}
$$

Of course, Eqs. G.4-G.6 automatically imply the expected conservation of three independent linear quantities: $K_4 = \quad v^2 + \frac{1}{2}v^4; K_5 = v^1 + \frac{1}{2}v^5$, and $K_6 = \frac{1}{2}v^6$. We note that these three conserved quantities must give rise to a Lie sub-algebra, because the product and sum of two conserved quantities remains a conserved quantity. Indeed, a direct computation shows that this is the case and that the resulting subalgebra is isomorphic to the rotation algebra $\mathfrak{so}(3)$. We also comment that the symmetry identified in the previous subsection is evidently preserved by these equations, which serves as a check for our computation of the geodesic equations. More concretely, the symmetry transformation takes G.1 and G.2 into each other, similarly for G.4 and G.5, and keep both G.3 and G.6 unchanged. In the next section, we further check that the quadratic quantities previously identified are conserved by our system of geodesic equations. After that, in Section 5, we finally solve the geodesic system to provide an explicit form for the geodesic flows in this system.

4 Conservation of the Quadratic Constants of Motion

In this section, we use the geodesic equations to check that the derivatives of each of our anticipated quadratic conserved quantities is indeed zero.

Energy H:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt} \quad 4v^1v^1 + 2v^1v^5 + 4v^2v^2 \quad 2v^2v^4 + 2v^3v^3 + \frac{1}{2}v^4v^4 + \frac{1}{2}v^5v^5 + \frac{1}{2}v^6v^6 \\
&= 8v^1\frac{dv^1}{dt} + 2 \quad v^1\frac{dv^5}{dt} + \frac{dv^1}{dt}v^5 \quad + 8v^2\frac{dv^2}{dt} \quad 2 \quad v^2\frac{dv^4}{dt} + \frac{dv^2}{dt}v^4 \quad + 4v^3\frac{dv^3}{dt} + v^4\frac{dv^4}{dt} + v^5\frac{dv^5}{dt} + v^6\frac{dv^6}{dt} \\
&= 2v^1 \quad 4\frac{dv^1}{dt} + \frac{dv^5}{dt} \quad + 2v^5 \quad \frac{dv^1}{dt} + \frac{1}{2}\frac{dv^5}{dt} \quad + 2v^2 \quad 4\frac{dv^2}{dt} \quad \frac{dv^4}{dt} \quad 2v^4 \quad \frac{dv^2}{dt} \quad \frac{1}{2}\frac{dv^4}{dt} \quad + v^6\frac{d}{dt}v^6 \\
&= 2v^1 \quad 8v^3v^1 + 4v^3v^5 \quad 4v^6v^2 \quad + 0 + 2v^2 \quad 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1 \quad 0 \\
&+ 2v^3 \quad 8v^1v^1 \quad 4v^1v^5 \quad 8v^2v^2 + 4v^2v^4 \quad + 0 = 0
\end{aligned}$$

where the second-to-last inequality is obtained by plugging in Eqs. G.1-G.6, and the last one simply by cancellation of the coefficients of all cubic terms.

Invariant Product Ω:

In this case, rather than plugging in the geodesic equations directly, we need to manipulate them to find the values of the derivatives $\frac{dv^i}{dt}$ individually. For a start, G.6 gives

$$\frac{dv^6}{dt} = 0; \tag{17}$$

and similarly G.3 directly implies

$$\frac{dv^3}{dt} = \quad 4v^1v^1 \quad 2v^1v^4 \quad 4v^2v^2 \quad 2v^2v^4: \tag{18}$$

Then, $\frac{dv^1}{dt}$ is found by substracting 2(G.5) from G.1 (and dividing by two). In a similar way, G.1 - 4(G.5) gives the value of $\frac{dv^5}{dt}$, G.2 + 2(G.4) leads to $\frac{dv^2}{dt}$ and G.2 - 4(G.4) to $\frac{dv^4}{dt}$. Thus we get

$$\frac{dv^1}{dt} = 4v^3v^1 + 2v^3v^5 \quad 2v^6v^2 \tag{19}$$

$$\frac{dv^2}{dt} = 4v^3v^2 \quad 2v^3v^4 + 2v^6v^1 \tag{20}$$

$$\frac{dv^4}{dt} = 8v^3v^2 \quad 4v^3v^4 + 2v^6v^1 \tag{21}$$

$$\frac{dv^5}{dt} = \quad 8v^3v^1 \quad 4v^3v^5 + 4v^6v^2: \tag{22}$$

Now we are ready to consider the derivative of Re[Ω]:

$$\frac{d}{dt}Re[\Omega] = \frac{d}{dt} \quad v^1v^4 + v^2v^5 + v^3v^6$$
$$= v^1\frac{dv^4}{dt} + \frac{dv^1}{dt}v^4 + v^2\frac{dv^5}{dt} + \frac{dv^2}{dt}v^5 + v^3\frac{dv^6}{dt} + \frac{dv^3}{dt}v^6$$

and plugging in the values of the derivatives computed above, we get

$$\frac{d}{dt}Re[\Omega] = v^1 \quad 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1 \quad + \quad 4v^3v^1 + 2v^3v^5 \quad 2v^6v^2 \quad v^4 + v^2 \quad 8v^3v^1 \quad 4v^3v^5 + 4v^6v^2$$
$$+ \quad 4v^3v^2 \quad 2v^3v^4 + 2v^6v^1 \quad v^5 + v^3\,(0) + \quad 4v^1v^1 \quad 2v^1v^4 \quad 4v^2v^2 \quad 2v^2v^4 \quad v^6 = 0$$

A similar computation goes through for the derivative of Im[Ω]:

$$\frac{d}{dt}Im[\Omega] = \frac{d}{dt} \quad 2v^1v^5 + 2v^2v^4 + 2v^3v^3 \quad \frac{1}{2}v^4v^4 \quad \frac{1}{2}v^5v^5 \quad \frac{1}{2}v^6v^6$$
$$= v^1\frac{dv^4}{dt} + \frac{dv^1}{dt}v^4 + v^2\frac{dv^5}{dt} + \frac{dv^2}{dt}v^5 + v^3\frac{dv^6}{dt} + \frac{dv^3}{dt}v^6$$
$$= \quad 2v^1 \quad 8v^3v^1 \quad 4v^3v^5 + 4v^6v^2 \quad 2 \quad 4v^3v^1 + 2v^3v^5 \quad 2v^6v^2 \quad v^5 + 2v^2 \quad 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1$$
$$+ 2 \quad 4v^3v^2 \quad 2v^3v^4 + 2v^6v^1 \quad v^4 + 4v^3 \quad 4v^1v^1 \quad 2v^1v^4 \quad 4v^2v^2 \quad 2v^2v^4 \quad v^4 \quad 8v^3v^2 \quad 4v^3v^4 + 2v^6v^1$$
$$+ v^5 \quad 8v^3v^1 \quad 4v^3v^5 + 4v^6v^2 \quad = 0$$

5 Solution of the Geodesic Equations

We begin by eliminating the variables $v^4; v^5; v^6$ from Eqs. G.1-G.3, leaving them in terms of $v^1; v^2; v^3$ and the constants $K_4; K_5; K_6$. First, focus on G.1, and substract 2*(G.5) from it to get

$$2\frac{dv^1}{dt} = 8v^3v^1 + 4v^3v^5 \quad 4v^6v^2$$
$$= 8v^3(v^1 + \frac{1}{2}v^5) \quad 4(v^6)v^2:$$

Then, divide throughout by 2, and substitute the conservation laws G.5 and G.6 of K_5 and K_6 for the terms in parenthesis:

$$\frac{dv^1}{dt} = 4K_5v^3 \quad 4K_6v^2:$$

Similarly, we can use G.4 and G.6 to simplify Eq. G.2

$$2\frac{dv^2}{dt} = 8v^3v^2 \quad 4v^3v^4 + 4v^6v^1 = \quad 8v^3(\quad v^2 + \frac{1}{2}v^4) + 4v^6v^1$$
$$! \quad \frac{dv^2}{dt} = \quad 4K_4v^3 + 4K_6v^1:$$

And, to simplify Eq. G.3, K_4 and K_5 can be substituted in as follows

$$\begin{aligned} 2\frac{dv^3}{dt} &= -8v^1v^1 - 4v^1v^5 - 8v^2v^3 + 4v^2v^4 \\ &= -8v^1(v^1 - \frac{1}{2}v^5) + 8v^2(-v^2 + \frac{1}{2}v^4) \\ \to \frac{dv^3}{dt} &= -4K_5v^1 + 4K_4v^2. \end{aligned}$$

The three resulting equations are Eqs. G.1'-G.3':

$$\begin{aligned} \frac{dv^1}{dt} &= -4K_6v^2 + 4K_5v^3. \\ \frac{dv^2}{dt} &= 4K_6v^1 - 4K_4v^3 \\ \frac{dv^3}{dt} &= -4K_5v^1 + 4K_4v^2. \end{aligned}$$

Note that these equations are effectively linear, thanks the fact that we could bring a constant K_i out as a common factor and coefficient for each of the terms. This is a consequence of the special symmetries of our product. In the general case, not all quadratic terms of the form v^iv^j can be reduced to linear terms, and this leads to more complicated solutions such as the Jacobi elliptic functions in the rigid body.

To solve our three equations above, we define the column vector $\vec{V} = (v^1, v^2, v^3)^T$, which allows us to rewrite them collectively in matrix form:

$$\frac{d}{dt}\vec{V} = M\vec{V}$$

where the antisymmetric matrix of coefficients is given by $M = \begin{pmatrix} 0 & -4K_6 & +4K_5 \\ +4K_6 & 0 & -4K_4 \\ -4K_5 & 4K_4 & 0 \end{pmatrix}.$

In general, the solutions for an equation of this form are determined by the exponential expression $e^{tM}\vec{C}$, for an arbitrary vector $\vec{C}$ of initial conditions. This exponential can be found through the Jordan canonical form of M, which in our case is the diagonal matrix D such that $D = T^{-1}MT$ for some non-singular linear transformation T. In particular, we will exploit the fact that $e^{tM} = T^{-1}e^{tD}T$.

It is a simple problem in linear algebra to find the eigenvalues of M, which are: $\lambda_1 = -4i\sqrt{K_4^2 + K_5^2 + K_6^2}$, $\lambda_2 = +4i\sqrt{K_4^2 + K_5^2 + K_6^2}$, and $\lambda_3 = 0$. These allow us to write D as:

$$D = \begin{pmatrix} -4i\sqrt{K_4^2 + K_5^2 + K_6^2} & 0 & 0 \\ 0 & 4i\sqrt{K_4^2 + K_5^2 + K_6^2} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

which implies

$$e^{tD} = \begin{pmatrix} e^{-4ti\sqrt{K_4^2 + K_5^2 + K_6^2}} & 0 & 0 \\ 0 & e^{4ti\sqrt{K_4^2 + K_5^2 + K_6^2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

On the other hand, some intricate algebra (or a computational calculation, for example on MatLab) renders the corresponding eigenvectors of M:

$$e_1 = \begin{pmatrix} -i \; K_4^2 K_5 + K_4 K_6 \sqrt{-K_4^2 - K_5^2 - K_6^2} + K_5^3 + K_5 K_6^2 \\ \dfrac{iK_4(K_4^2 + K_5^2 + K_6^2) - K_5 K_6 \sqrt{K_4^2 + K_5^2 + K_6^2}}{(K_4^2 + K_5^2)\sqrt{K_4^2 + K_5^2 + K_6^2}} \end{pmatrix}$$

$$e_2 = \begin{pmatrix} i \; K_4^2 K_5 + K_4 K_6 \sqrt{-K_4^2 - K_5^2 - K_6^2} - K_5^3 - K_5 K_6^2 \\ \dfrac{iK_4(K_4^2 + K_5^2 + K_6^2) - K_5 K_6 \sqrt{K_4^2 + K_5^2 + K_6^2}}{(K_4^2 + K_5^2)\sqrt{K_4^2 + K_5^2 + K_6^2}} \end{pmatrix}$$

$$e_3 = \begin{pmatrix} K_4 \\ K_5 \\ K_6 \end{pmatrix};$$

which are the three columns of the diagonalizing transformation T. In this way, they determine the solutions through

$$e^{tM}\vec{C} = T^{-1}e^{tD}T\vec{C}. \tag{23}$$

Our result comes with the surprise of an additional linear constant of motion, some intricate linear combination of the v^4, v^5, v^6 whose existence is implied by the zero eigenvalue $\lambda_3 = 0$ of D. In other words, in the system of coordinates that diagonalizes M into D, the third coordinate is conserved. Nonetheless, we note that this one is not independent from the six conserved quantities we discussed previously (we cannot have seven independent constants in a six-dimensional system), so that the total number of independent conserved quantities remains unchanged by this observation.

The explicit form of the solutions is in general complicated, as a consequence of the complex form of the e_i columns in the diagonalizing transformation matrix. But they can be evaluated numerically in a direct way, given the initial conditions. Consequently, rather than presenting the final general solution in vector form, we prefer the neater option of supplementing Eq. 23 with a script of Matlab code that carries out the matrix products in said equation and outputs the resulting solutions in text format (see Appendix A). Finally, these solutions for v^4, v^5, v^6 can then be used to find v^1, v^2, v^3 from Eqs. G.4-G.6, completing out our computation. Qualitatively, the resulting solutions are given by linear combinations of exponential functions with exponents that are linear in t. The coefficients in the exponent are imaginary and depend on the initial conditions. Therefore, our solutions trace out sinusoidal oscillations of the coordinates with time; with the exception of the coordinate v^6, which is of course constant as we previously found. Therefore, these solutions are indeed analogous to the trigonometric solutions of the partially-symmetric rigid body, as predicted. We conclude this section by providing in Fig. 5 a plot of an example geodesic trajectory computed (throught the first script in Appendix A) for the particular choice of initial conditions: $k_1 = 20, k_2 = 10, k_3 = 10, k_4 = 50, k_5 = 10, k_6 = -10$, in arbitrary units that out of habit we decide to label meters or (m).

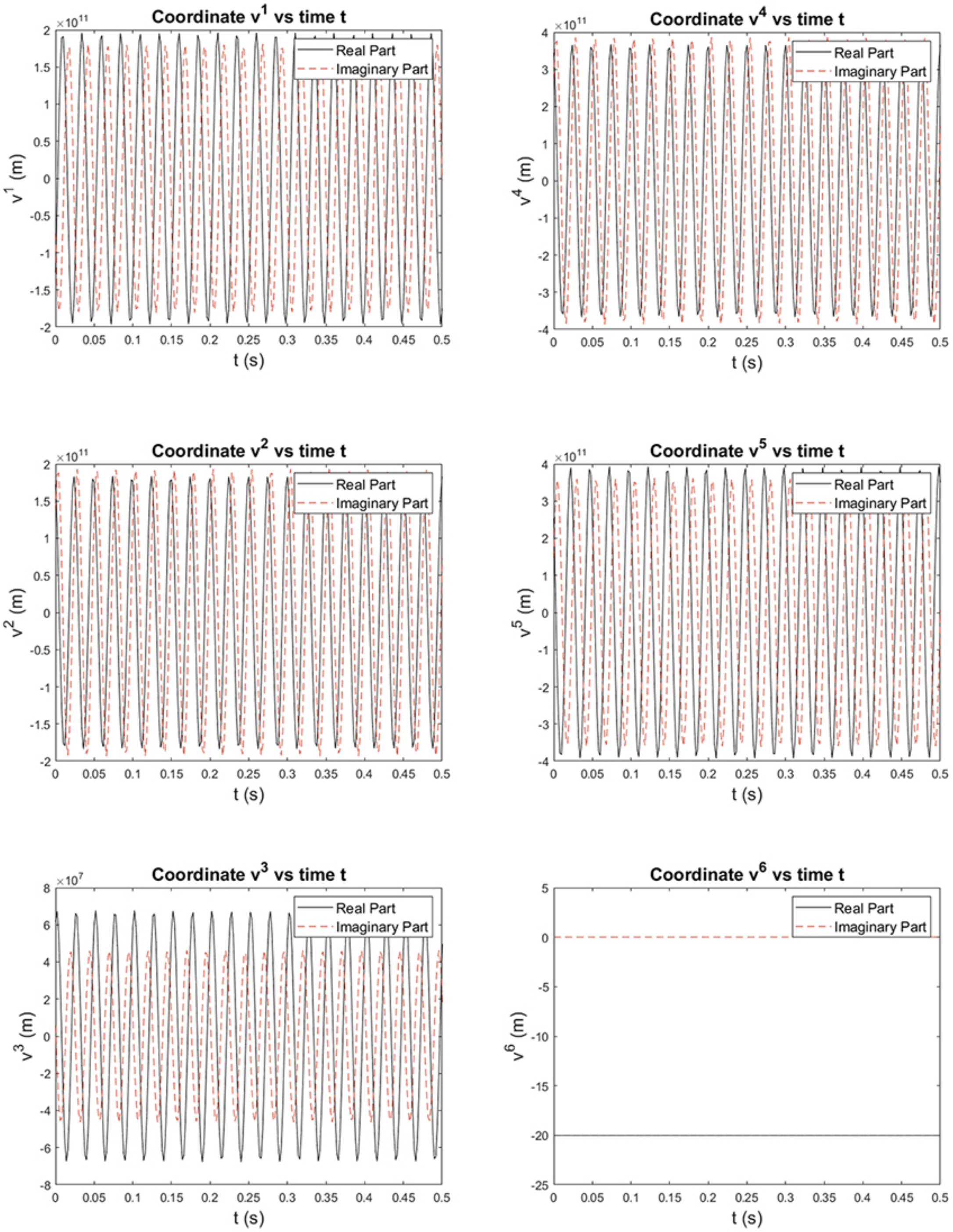

Figure 1: Example geodesic trajectory for timestep of $\Delta t = 0{:}002$ (s), and initial conditions: $k_1 = 20$ (m); $k_2 = 10$ (m); $k_3 = 10$ (m); $k_4 = 50$ (m); $k_5 = 10$ (m); $k_6 = \;\; 10$ (m). Note the different axis scales: in particular, v^6 is by far the smallest in magnitude, followed by v^3.

6 Conclusions

Equation 23 describes the geodesic flows on the Lorentz group of relativistic transformations, consisting of combinations of rotations and boosts. As such, they can be interpreted as the paths that connect two relativistic frames of references, each with their own choice of axis orientations and constant velocity vector, under no additional constrains except minimizing the distance determined by the metric obtained from our chosen product (11). Unfortunately, the physical interpretation of this notion of distance is not evident.

From a mathematical point of view, the Lorentz geodesics represent a dynamical system analogous to the rigid body with in-plane rotational symmetry (that is, with a single component of angular momentum conserved), except that both the number of degrees of freedom and that of constants of motion are increased by three when the Lorentz boosts are added to the group of three-dimensional spatial rotations. To obtain this analogy, who choose the positive metric to be as symmetric as possible. An alternative perspective is to think of the geodesics on the Lorenz group as a finite-dimensional case of ideal fluids as geodesics of the infinite-dimensional group of incompressible vector fields.

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Appendix A

In this section I present the MatLab scripts that I used as assistance to my calculations for this project. First, this MatLab script computes the geodesic solutions (and plots them as presented in Fig. 5):

```
%THIS SCRIPT COMPUTES MATRIX PRODUCTS TO FIND THE VECTOR FORM OF THE GEODESIC
%SOLUTIONS

%Initialize variables for arbitrary initial conditions
syms k4
syms k5
syms k6
syms t
syms k1
syms k2
syms k3

%Transformation matrix
T = [i*(k4^2*k5+k4*k6*sqrt(-k4^2-k5^2-k6^2)+k5^3+k5*k6^2), ...
     i*(k4^2*k5+k4*k6*sqrt(-k4^2-k5^2-k6^2)+k5^3+k5*k6^2), k4; ...
     -i*k4*(k4^2+k5^2+k6^2)-k5*k6*(sqrt(k4^2+k5^2+k6^2)), ...
     i*k4*(k4^2+k5^2+k6^2)-k5*k6*(sqrt(k4^2+k5^2+k6^2)), k5; ...
     (k4^2+k5^2)*sqrt(k4^2+k5^2+k6^2),...
     (k4^2+k5^2)*sqrt(k4^2+k5^2+k6^2), k6]
Tt = transpose(T)

%Original system of equations
M = [0, -4*k6, +4*k5; +4*k6, 0, -4*k4; -4*k5, 4*k4, 0]
%Find eigenvalues and eigenvectors
[eve, eva] = eig(M)

%Undo diagonalization of E = exp(D)
E = [exp(-4*t*i*sqrt(k4^2+k5^2+k6^2)), 0, 0; ...
     0, exp(4*t*i*sqrt(k4^2+k5^2+k6^2)), 0; 0, 0, 0]
Res = Tt*E*T

%Apply to ICs
ResWIC = Res*[k1, k2, k3]'

%EXAMPLE PLOT
%Define ICs
k4 = 50;
k5 = 10;
k6 = -10;
k1 = 20;
k2 = 10;
```

```
k3 = 10;

%Re-initialize variables
M2 = [0, -4*k6, +4*k5; +4*k6, 0, -4*k4; -4*k5, 4*k4, 0];
T2 = [i*(k4^2*k5+k4*k6*sqrt(-k4^2-k5^2-k6^2)+k5^3+k5*k6^2), ...
    i*(k4^2*k5+k4*k6*sqrt(-k4^2-k5^2-k6^2)+k5^3+k5*k6^2), k4; ...
    -i*k4*(k4^2+k5^2+k6^2)-k5*k6*(sqrt(k4^2+k5^2+k6^2)), ...
    i*k4*(k4^2+k5^2+k6^2)-k5*k6*(sqrt(k4^2+k5^2+k6^2)), k5; ...
    (k4^2+k5^2)*sqrt(k4^2+k5^2+k6^2),...
    (k4^2+k5^2)*sqrt(k4^2+k5^2+k6^2), k6];
[eve2, eva2] = eig(M2);
Tt2 = transpose(T2);

%Compute and plot
npoints = 50000
for t=1:npoints
    E2 = [exp(-4*t*i*sqrt(k4^2+k5^2+k6^2)), 0, 0; ...
        0, exp(4*t*i*sqrt(k4^2+k5^2+k6^2)), 0; 0, 0, 0];
    Res = Tt2*E2*T2;
    ResWIC2 = Res*[k1, k2, k3]';
    p1(t) = ResWIC2(1);
    p2(t) = ResWIC2(2);
    p3(t) = ResWIC2(3);
end

p4 = 2*p2+k4;
p5 = -2*p1+k5;
p6 = zeros(npoints, 1);
p6(:) = 2*k6;

figure(1)
plot(linspace(0,100,50000), real(p1), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^1 (m)", 'FontSize', 15)
hold on;
plot(linspace(0,100,50000), imag(p1),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
title("Coordinate v^1 vs time t", 'FontSize', 15)
hold off;

figure(2)
plot(linspace(0,100,50000), real(p2), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^2 (m)", 'FontSize', 15)
```

```
hold on;
plot(linspace(0,100,50000), imag(p2),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
title("Coordinate v^2 vs time t", 'FontSize', 15)
hold off;

figure(3)
plot(linspace(0,100,50000), real(p3), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^3 (m)", 'FontSize', 15)
hold on;
plot(linspace(0,100,50000), imag(p3),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
title("Coordinate v^3 vs time t", 'FontSize', 15)
hold off;

figure(4)
plot(linspace(0,100,50000), real(p4), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^4 (m)", 'FontSize', 15)
hold on;
plot(linspace(0,100,50000), imag(p4),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
title("Coordinate v^4 vs time t", 'FontSize', 15)
hold off;

figure(5)
plot(linspace(0,100,50000), real(p5), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^5 (m)", 'FontSize', 15)
hold on;
plot(linspace(0,100,50000), imag(p5),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
title("Coordinate v^5 vs time t", 'FontSize', 15)
hold off;

figure(6)

plot(linspace(0,100,50000), real(p6), 'LineWidth', 0.2, 'Color',[0,0,0]);
xlabel("t (s)", 'FontSize', 15)
ylabel("v^6 (m)", 'FontSize', 15)
```

```
hold on;
plot(linspace(0,100,50000), imag(p6),'--' , 'LineWidth', 0.2, 'Color',[0.8,0.2,0.1]);
legend(["Real Part", "Imaginary Part"], 'FontSize', 12);
xlim([0, 1])
ylim([-25, 5])
title("Coordinate v^6 vs time t", 'FontSize', 15)
hold off;

% --- END
```

We also include the MatLab script used to compute the subalgebra of linear constants of motion:

```
%THIS SCRIPT COMPUTES THE SUBALGEBRA OF CONSERVED QUANTITIES

m4 = [0, -1i/2; -1i/2, 0];
m5 = [0, -1/2; 1/2, 0];
m6 = [-1i/2, 0; 0, 1i/2];
m1 = [0, -2; 0, 0];
m2 = [0, 2i; 0, 0];
m3 = [-1, 0; 0, 1];

%Conserved quantities
k4 = -m2+1/2*m4
k5 = m1+1/2*m5
k6 = 1/2*m6; % *(sqrt(2/9))

%Commutators with each basis matrix
c41 = k4*m1-m1*k4
c42 = k4*m2-m2*k4
c43 = k4*m3-m3*k4
c44 = k4*m4-m4*k4
c45 = k4*m5-m5*k4
c46 = k4*m6-m6*k4

c51 = k5*m1-m1*k5
c52 = k5*m2-m2*k5
c53 = k5*m3-m3*k5
c54 = k5*m4-m4*k5
c55 = k5*m5-m5*k5
c56 = k5*m6-m6*k5

%Commutators of conserved quantities
comm45 = k4*k5-k5*k4
comm46 =  k4*k6-k6*k4
comm56 =  k5*k6-k6*k5

% --- END
```

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