Van der Corput's method for exponential sums and the Divisor Problem

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1 Introduction

The Dirichlet Divisor Problem, named after the German mathematician Peter Gustav Lejeune Dirichlet, is a classical problem in number theory. It concerns the distribution of the number of divisors of positive integers, which plays a crucial role in various areas of mathematics, including analytic number theory and algebraic geometry. The problem can be succinctly stated as follows: given a positive integer n, let the divisor function $d(n)$ be the number of positive integer divisors of n (including 1 and n itself). From now on, we only deal with positive integers unless otherwise specified. For example, 24 has divisors $d = 1, 2, 3, 4, 6, 8, 12, 24$. Thus, $d(24) = 8$. If we let $D(x) = \sum_{n \leq x} d(n)$, then $D(x)$ sums the numbers of divisors of integers less than or equal to x. Thus, the average number of divisors for any integer $\leq x$ is $D(x)/x$. In 1849, Dirichlet proposed the question of the size of $D(x)$ and developed a method, now known as the hyperbola method, to prove that $D(x) = \sum_{n \leq x} d(n) = x \log x + x(2\gamma - 1) + O(\sqrt{x})$, where $\gamma = .577215\ldots$ is Euler's constant. His proof will be shown in section 3. The problem of finding the best error term in the expression for $D(x)$ is now known as the Dirichlet divisor problem. If we write the error as $O(x^{\theta+\epsilon})$, where ϵ is any positive number, it is conjectured that this holds for $\theta = 1/4$. In fact, in 1916, G. H. Hardy showed that no θ less than $1/4$ works.

It turns out the error term is closly connected to the estimated of exponential sums. In this paper, we will introduce van der Corput's method of exponential sums, which provides a systematic way to bound exponential sums by exploiting the oscillatory behavior of the complex exponential function. We will describe van der Corput's method as well as its application to the Dirichlet divisor problem.

2 Notation

In this section, we introduce some notation that we will use in this thesis.
1. $\lfloor x \rfloor$: $\lfloor x \rfloor$ is defined to be the largest integer that does not exceed x.
Examples: $\lfloor 1.5 \rfloor = 1$, $\lfloor 2 \rfloor = 2$.

2. $\{x\}$: $\{x\}$ is defined to be the fractional part of x. $\{x\} = x - \lfloor x \rfloor$. Notice that for any x, $\{x\} < 1$.
Examples: $\{1.8\} = 0.8$, $\{2\} = 0$.

3. Big O notation: Let f, the function to be estimated, be a real or complex valued function, and let g, the comparison function, be a real valued positive function. Let both functions be defined on some unbounded subset of the positive real numbers. If there exists a real number x_0 and a positive real constant c such that $|f(x)| \leq cg(x)$ for all $x \geq x_0$, then we write $f(x) = O(g(x))$. Equivalently, we can write $f(x) \ll g(x)$.
Examples: As $x \to \infty$,

$$3x^2 + x \ll x^2,$$

$$3x^2 + x \ll x^2 \log x.$$

4. Small o notation: For two functions $f(x)$ and $g(x)$, we write $f(x) = o(g(x))$ as $x \to c$ if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 0.$$

5. Let f be a a real function. We write $f \approx g$ if and only if $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be in $C^1(\mathbb{R})$ if it is continuously differentiable, i.e., both f and its derivative f' exist and are continuous on $\mathbb{R}$. Similarly, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be in $C^2(\mathbb{R})$ if it is twice continuously differentiable, i.e., both f and its first two derivatives f' and f'' exist and are continuous on $\mathbb{R}$. We similarly define $C^1([a, b])$ and $C^2([a, b])$ for functions $f : [a, b] \to \mathbb{R}$.

7. An integrable function $f : \mathbb{R} \to \mathbb{C}$ is said to be in $L^1(\mathbb{R})$ if the integral of the absolute value of $f(x)$ over the entire real line exists and is finite. In other words, a function f belongs to $L^1(\mathbb{R})$ if

$$\int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$$

8. $\|t\|$: Let t be a rational number. We use $\|t\|$ to denote the distance from t to the closest integer.

3 Dirichlet's Method

Dirichlet's method turns the problem of estimating $D(N)$ into the problem of counting the lattice points in a bounded region.

Theorem 1.1: For any real number $N \geq 1$, we have $\sum_{n \leq N} d(n) = N \log N + N(2\gamma - 1) + O(\sqrt{N})$.

Proof. Notice that geometrically, $d(n)$ counts the number of lattice points (points with integer coordinates) on the parabola $xy = n$. Thus, $D(N)$ counts the lattice points in the

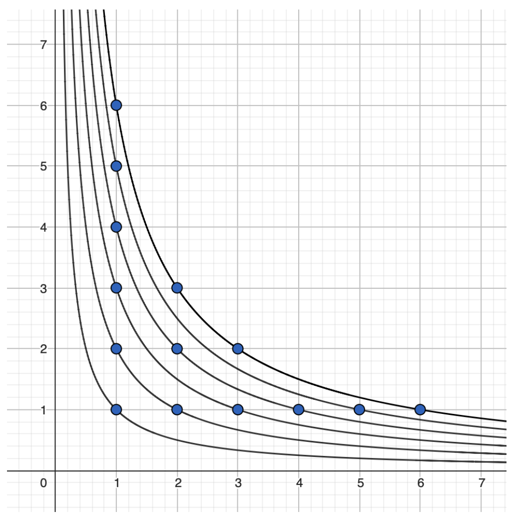
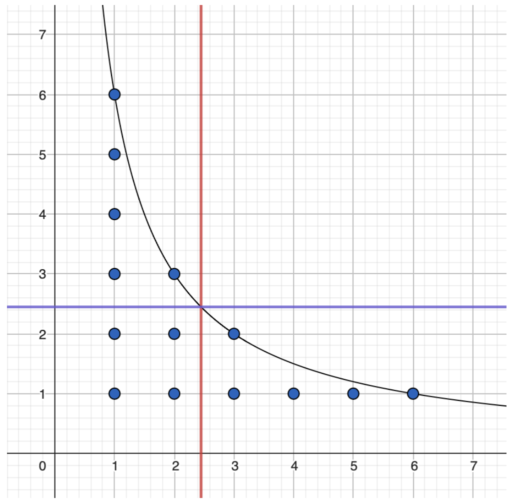


Figure 1: lattice points under xy=6

first quadrant that are on or below the parabola $xy = N$. We divide this region into two parts: the region bounded by $xy = N$, the positive x and y axes, and the line $x = \sqrt{N}$, and the region bounded by $xy = N$, the positive x and y axes, and the line $y = \sqrt{N}$. Observe that there is a square of side length $\sqrt{N}$ common to the two regions. In the region bounded by $xy = N$ and $x = \sqrt{N}$, there are $\lfloor \frac{N}{n} \rfloor$ lattice points on each line $x = n$ with $n \leq \sqrt{N}$. There are the same number of lattice points in our other region. Thus, adding the number of lattice points in both regions and subtracting the number in the square (because they are



counted twice), we see that the number of lattice points under the hyperbola is

$$\begin{aligned}\sum_{n=1}^{N} d(n) &= 2\Big(\sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \lfloor\frac{N}{n}\rfloor\Big) - \lfloor\sqrt{N}\rfloor^2 \\ &= 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \Big(\frac{N}{n} - \{\frac{N}{n}\}\Big) - (\sqrt{N} - \{\sqrt{N}\})^2 \qquad (1) \\ &= 2N\sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \frac{1}{n} - 2\sum_{n=1}^{\lfloor\sqrt{N}\rfloor} \{\frac{N}{n}\} - N + 2\sqrt{N}\{\sqrt{N}\} - \{\sqrt{N}\}^2.\end{aligned}$$

Noticing that the fractional part is always < 1, we find that

$$2\sum_{n=1}^{\sqrt{N}} \{\frac{N}{n}\} < 2\sum_{n=1}^{\sqrt{N}} 1 = 2\sqrt{N} = O(\sqrt{N}).$$

Also,

$$2\sqrt{N}\{\sqrt{N}\} < 2\sqrt{N} = O(\sqrt{N}).$$

Finally,

$$\{\sqrt{N}\}^2 < 1,$$

which is absorbed by the $O(\sqrt{N})$. Hence, combining these terms, we obtain that (1) is

$$= 2N\sum_{n=1}^{\sqrt{N}} \frac{1}{n} - N + O(\sqrt{N}). \qquad (2)$$

To estimate the sum on the right, we quote the following well known result.

Theorem 1.2 (Partial Sums of Harmonic Series):
For $N \geq 1$,

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \gamma + \frac{1}{2N} + O(N^{-2}), \tag{3}$$

where $\gamma = 0.57721\ldots$ is Euler's constant.

This can be derived from the Euler-Maclaurin sum formula.

Using **Theorem 1.2** in (2), we have

$$\begin{aligned}\sum_{n=1}^{N} d(n) &= 2N(\ln\sqrt{N} + \gamma + O(\frac{1}{\sqrt{N}})) - N + O(\sqrt{N}) \\ &= 2N\ln\sqrt{N} + 2N\gamma + O(\frac{N}{\sqrt{N}}) - N + O(\sqrt{N}) \\ &= 2N\ln N^{\frac{1}{2}} + 2N\gamma - N + O(\sqrt{N}) \\ &= N\ln N + N(2\gamma - 1) + O(\sqrt{N}).\end{aligned} \tag{4}$$

This completes the proof. □

4 Van der Corput's Method

Johannes van der Corput was a Dutch mathematician who worked in the field of analytic number theory. He introduced the method of exponential sums which provided a new tool in number theory. In 1922, van der Corput used his method to show that the remainder term in the Dirichlet divisor problem has order $\ll_\epsilon x^{33/100+\epsilon}$.

We start by defining exponential sums.

Definition 1 (Exponential sum). *Let $x_1, x_2, ..., x_N$ be real numbers. An exponential sum is a sum of the form*

$$\sum_{i=1}^{N} e(x_i),$$

where we write $e(x)$ to denote $e^{2\pi ix}$.

Notice that since $|e(x)| = 1$ for real numbers x, we have that $|\sum_{i=1}^{N} e(x_i)| \leq N$, with equality whenever the terms are all equal.

We will make use of the Poisson summation formula to study trigonometric sums.

Definition 2 (Fourier transform)**.** *Let $f \in L^1(\mathbb{R})$. The Fourier transform of f is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ given by*

$$\hat{f}(\theta) = \int_{-\infty}^{\infty} f(t)e(-\theta t).$$

Theorem 2.1: Let $f \in L^1(\mathbb{R})$. Assume that the series

$$\varphi(t) = \sum_{n \in \mathbb{Z}} f(n+t) \tag{5}$$

converges for all t and its sum defines a function of bounded variation on [0,1] that is continuous at 0. Then we have

$$\lim_{N \to \infty} \sum_{|\nu| \leq N} \hat{f}(\nu) = \sum_{n \in \mathbb{Z}} f(n). \tag{6}$$

Theorem 2.2: Let $f \in C^1([a,b])$ be such that $f'(t)$ is monotone and of constant sign on $[a,b]$. Suppose that for $a \leq t \leq b$ we have

$$|f'(t)| \geq m > 0.$$

Then

$$|\int_a^b e(f(t))dt| \leq \frac{2}{\pi m}. \tag{7}$$

Proof. Without loss of generality we may assume f' is non-increasing on (a,b). We have

$$\frac{de(f(t))}{dt} = e(f(t))2\pi i f'(t).$$

Thus,

$$\begin{aligned}
\left|2\pi \int_a^b e(f(t))dt\right| &= \left|\int_a^b \frac{1}{f'(t)} de(f(t))\right| \\
&= \left|\left[\frac{e(f(t))}{f'(t)}\right]_a^b - \int_a^b e(f(t))d\frac{1}{f'(t)}\right| \\
&= \left|\frac{e(f(b))}{f'(b)} - \frac{e(f(a))}{f'(a)} - \int_a^b e(f(t))d\frac{1}{f'(t)}\right| \\
&\leq \left|\frac{e(f(b))}{f'(b)}\right| + \left|\frac{e(f(a))}{f'(a)}\right| + \int_a^b \left|d\frac{1}{f'(t)}\right| \\
&\leq \frac{2}{m} + \left|\int_a^b d\frac{1}{f'(t)}\right|
\end{aligned} \tag{8}$$

where the last line follows from the fact that $d(1/f'(t))$ is of one sign on $[a,b]$. Hence, the above is

$$\leq 2/m + \left|\frac{1}{f'(b)}\right| + \left|\frac{1}{f'(a)}\right| \leq \frac{2}{m}.$$

□

Theorem 2.3: Let $f \in C^2[a,b]$ be such that $f''(t)$ has constant sign on $[a,b]$. Suppose that for $a \leq t \leq b$ we have $|f''(t)| \geq r > 0$. Then we have

$$\left|\int_a^b e(f(t))dt\right| \leq 4\sqrt{\frac{2}{\pi r}}. \tag{9}$$

Proof. Let us suppose, without loss of generality, that $f''(t) \leq -r \leq 0$ for $a \leq t \leq b$. Then $f'(t)$ vanishes **at most** once on $[a,b]$, say at $t = c$, that is, $f'(c) = 0$. Then, we can separate $[a,b]$ into three intervals and write

$$I := \int_a^b e(f(t))dt = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b} = I_1 + I_2 + I_3,$$

where the positive parameter δ satisfies $a + \delta \leq c \leq b - \delta$. By Fundamental Theorem of Calculus, we have

$$|f'(t) - f'(c)| = |f'(t)| = \left|\int_c^t f''(\nu)d\nu\right| \geq r|t-c| \geq r\delta$$

for $t \in [a, c-\delta] \cup [c-\delta, b]$. On $[a, c-\delta]$ and $[c+\delta, b]$ respectively, $f'(t)$ is decreasing and of constant sign. Thus we can apply Theorem 2.2 and obtain

$$|I_1| + |I_3| \leq \frac{4}{\pi r\delta}.$$

Since trivially, $|I_2| \leq 2\delta$, it follows that

$$|I| \leq 2\delta + \frac{4}{\pi r\delta}.$$

By choosing $\delta = \sqrt{\frac{2}{\pi r}}$, we have the stated result.

If with this choice of δ we have either $c < a+\delta$ or $c > b-\delta$, say $c < a+\delta$, then we can write:

$$|I| \leq \left|\int_a^c\right| + \left|\int_c^{c+\delta}\right| + \left|\int_{c+\delta}^b\right| \leq 2\delta + \frac{2}{\pi r\delta},$$

so that the stated upper bound remains valid.

If $f'(t)$ does not vanish on $[a,b]$, then $f'(t)$ is decreasing and of a constant sign on $[a,b]$. Without loss of generality, suppose $f'(t) > 0$. Then, $\inf_{a<t<b} f'(t) = f'(b) > 0$. For $t \in [a, b-\delta]$,

$$|f'(t) - f'(b)| = f'(t) - f'(b) = \left| \int_t^b f''(\nu) d\nu \right| \geq r|t-b| \geq r\delta$$

Thus, $f'(t) > r\delta + f'(b) > r\delta$. Hence, by Theorem 2.2, $\int_a^{b-\delta} e(f(t))dt \leq 4/\pi r\delta$. Also, we trivially have that $|\int_{b-\delta}^b| \leq \delta$. Thus,

$$|I| \leq \left| \int_a^{b-\delta} \right| + \left| \int_{b-\delta}^b \right| \leq \frac{4}{\pi r \delta} + \delta.$$

Choosing $\delta = \frac{2}{\sqrt{\pi r}}$, we obtain a bound of $\leq \frac{2}{\sqrt{\pi r}}$, so that (9) again holds. □

Lemma 2.4: Let $0 \leq M < N$ and ν be integers, and let t be a real number. Then we have

$$\sum_{M<\nu\leq N} e(\nu t) \ll \min(N-M, \frac{1}{\|t\|}),$$

where $\|t\|$ denotes the distance between t and its nearest integer.

Proof. The formula for the sum of a geometric series gives

$$\begin{aligned} \sum_{M<\nu\leq N} e(\nu t) &= e((M+1)t)(1 + e(t) + \cdots + e(t(N-1))) \\ &= e((M+1)t) \frac{1 - e(tN)}{1 - e(t)} \qquad (10) \\ &= e((M+1)t) \frac{e(\frac{Nt}{2})\big(e(-\frac{Nt}{2}) - e(\frac{Nt}{2})\big)}{e(\frac{t}{2})\big(e(-\frac{t}{2}) - e(\frac{t}{2})\big)}. \end{aligned}$$

Thus,

$$\begin{aligned} \Big| \sum_{M<\nu\leq N} e(\nu t) \Big| &= \left| \frac{e(-\frac{Nt}{2}) - e(\frac{Nt}{2})}{e(-\frac{t}{2}) - e(\frac{t}{2})} \right| \\ &= \left| \frac{\sin(\pi N t)}{\sin(\pi t)} \right| \qquad (11) \\ &\leq \frac{1}{|\sin(\pi t)|}. \end{aligned}$$

For $t \in [-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]$, we have $|\sin \pi t| \geq 2|t|$. Since $0 < \|t\| \leq 1/2$, $\sin(\pi t) \geq 2\|t\|$. Hence, $\frac{1}{\sin(\pi t)} \ll \frac{1}{\|t\|}$. When t gets close to an integer, $\frac{1}{\|t\|}$ gets large. In that case we can use the trivial bound, $\sum_{M<\nu\le N} e(-\nu t) \ll N-M$. Thus, we obtain the stated result. $\square$

Theorem 2.5: Let $f \in C^1([a,b])$ be such that $f'(t)$ is monotone on $[a,b]$. Set

$$\alpha := \inf_{a<t<b} f'(t), \quad \beta := \sup_{a<t<b} f'(t).$$

Then, for each $\epsilon > 0$, we have

$$\sum_{a<n\le b} e(f(n)) = \sum_{\alpha-\epsilon<\nu<\beta+\epsilon} \int_a^b e(f(t)-\nu t)dt + O_\epsilon(\log(\beta-\alpha+2)). \tag{12}$$

Proof. Let ϵ be fixed. If $f(t)$ is replaced by $g(t) = f(t)+kt$ for any $k \in \mathbb{Z}$, the left-hand side of (12) is clearly invariant. If we write

$$\alpha' := \inf_{a<t<b} g'(t) = \inf_{a<t<b} f'(t) + k = \alpha + k$$

and

$$\beta' := \sup_{a<t<b} g'(t) = \sup_{a<t<b} f'(t) + k = \beta + k,$$

and set $\mu = \nu - k$, then the right-hand side equals

$$\sum_{\alpha'-\epsilon<\mu<\beta'+\epsilon} \int_a^b e(f(t)-\mu t)dt + O_\epsilon(\log(\beta'-\alpha'+2)),$$

and the O-term equals $O_\epsilon(\log(\beta-\alpha+2))$, Thus, we may translate the sum in (12) and assume that $-1 \le \alpha - \epsilon < 0$.

We can equally well restrict ourselves to the case where a and b are of the form $n+\frac{1}{2}, m+\frac{1}{2}$ for some $n, m \in \mathbb{Z}$. To see this, note that for any b, there exists an integer m such that $|b-m| \le \frac{1}{2}$. If we replace b by $m+\frac{1}{2}$, then the left-hand side of (12) becomes

$$\sum_{a<n\le m+1/2} e(f(n)),$$

which changes the original sum on the left-hand side of (12) by at most $O(1)$. What about the right-hand side? Without loss of generality, let us assume that $m \le b < m+1/2$. If we replace b by $m+\frac{1}{2}$, the error resulting in the right-hand side of (12) is at most

$$\begin{aligned}
&\left|\sum_{\alpha-\epsilon<\nu<\beta+\epsilon}\int_b^{m+\frac{1}{2}} e(f(t)-\nu t)dt\right| \\
&=\left|\int_b^{m+\frac{1}{2}} e(f(t))\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t)dt\right| \\
&\leq \int_b^{m+\frac{1}{2}}\left|\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t)\right|dt \\
&\leq \int_m^{m+\frac{1}{2}}\left|\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t)\right|dt,
\end{aligned}$$

since $m\leq b$ by assumption. Now by Lemma 2.4,

$$\sum_{\alpha-\epsilon<\nu<\beta+\epsilon} e(-\nu t)\ll \min\Big(\beta-\alpha+2,\frac{1}{\|t\|}\Big),$$

since $\alpha-\epsilon$ and $\beta+\epsilon$ are not necessarily integers. Hence, the expression above is

$$\begin{aligned}
&\ll \int_m^{m+\frac{1}{\beta-\alpha+2}}(\beta-\alpha+2)dt+\int_{m+\frac{1}{\beta-\alpha+2}}^{m+\frac{1}{2}}\frac{1}{\|t\|}dt \\
&\ll 1+\log(\beta-\alpha+2)\ll\log(\beta-\alpha+2).
\end{aligned}$$

By a similar argument, we may replace a by an integer $n+\frac{1}{2}$. Thus, the error on the right hand side is $O(log(\beta-\alpha+2))$.

Continuing with the argument, since$f'(t)$ is monotone on $[a,b]$, we may suppose f' is decreasing on $[a,b]$ without loss of generality. Let us set

$$F(t)=\begin{cases} e(f(t)), & \text{if } a<t\leq b \\ 0, & \text{otherwise}\end{cases}$$

Let $\varphi:=\sum_{n\in\mathbb{Z}}F(n+t)$. Then φ is continuous at 0 since $a,b\notin\mathbb{Z}$. Moreover, φ has bounded variation on $[0,1]$. The Poisson formula (5) then implies that

$$\sum_{a<n\leq b} e(f(n))=\sum_{|\nu|\leq N}\hat{F}(\nu)+o(1)\qquad (N\to\infty)$$

with

$$\hat{F}(\nu) = \int_a^b e(f(t) - \nu t)dt.$$

Taking account of our assumption that $-1 \leq \alpha - \epsilon < 0$, it suffices to show that we have

$$\sum_{\substack{|\nu|\leq N \\ \nu\notin[0,\beta+\epsilon]}} \hat{F}(\nu) = O_\epsilon(\log(\beta+2)). \tag{13}$$

For then, (12) clearly holds. Since

$$\frac{d}{dt}e(f(t)-\nu t) = e(f(t)-\nu t)2\pi i(f'(t)-\nu),$$

we see that

$$\begin{aligned} 2\pi i\hat{F}(\nu) &= \int_a^b \frac{d\{e(f(t)-\nu t)\}}{f'(t)-\nu} \\ &= \left[\frac{e(f(t)-\nu t)}{f'(t)-\nu}\right]_a^b - \int_a^b e(f(t)-\nu t)d\{\frac{1}{f'(t)-\nu}\} \\ &= \frac{e(f(b)-\nu b)}{f'(b)-\nu} - \frac{e(f(a)-\nu a)}{f'(a)-\nu} - \int_a^b e(f(t)-\nu t)d\{\frac{1}{f'(t)-\nu}\} \end{aligned} \tag{14}$$

As a, b are of the form $m+\frac{1}{2}$, with m$\in\mathbb{Z}$,

$$\begin{aligned} e(f(b)-\nu b) &= e(f(b))e(-\nu b) \\ &= e(f(b))e(-\nu m - \tfrac{1}{2}\nu) \\ &= e(f(b))e^{-2\pi i m\nu}e^{-\pi i\nu} \\ &= e(f(b))\cdot 1\cdot (e^{-\pi i})^\nu \\ &= e(f(b))(-1)^\nu \end{aligned} \tag{15}$$

Similarly, $e(f(a)-\nu a) = e(f(a))(-1)^\nu$. Moreover, by assumption, $f'(t)$ is decreasing on $[a,b]$. Thus, $f'(b) = \inf_{a<t<b} f'(t) = \alpha$ and $f'(a) = \sup_{a<t<b} f'(t) = \beta$. We may therefore rewrite (14) as

$$\begin{aligned} 2\pi i\hat{F}(\nu) &= (-1)^\nu\frac{e(f(b))}{\alpha-\nu} + (-1)^{\nu+1}\frac{e(f(a))}{\beta-\nu} - \int_a^b e(f(t)-\nu t)d\{\frac{1}{f'(t)-\nu}\} \\ &= (-1)^\nu\frac{e(f(b))}{\alpha-\nu} + (-1)^{\nu+1}\frac{e(f(a))}{\beta-\nu} + O(\frac{1}{\alpha-\nu} - \frac{1}{\beta-\nu}). \end{aligned} \tag{16}$$

The contribution to (13) of the first two terms is $O_\epsilon(1)$ since their sums telescope. The contribution of the error term is

$$
\begin{aligned}
&\sum_{\nu\notin[0,\beta+\epsilon]} \frac{1}{\alpha-\nu} - \frac{1}{\beta-\nu} \\
&= \sum_{\nu\notin[0,\beta+\epsilon]} \frac{\beta-\alpha}{(\alpha-\nu)(\beta-\nu)} \\
&\ll \sum_{\nu\notin[0,\beta+\epsilon]} \frac{\beta+1}{\nu(\nu-\beta)} \qquad \text{(as } -1+\epsilon \le \alpha < \epsilon) \\
&= \sum_{\nu\le -1} \frac{\beta+1}{\nu(\nu-\beta)} + \sum_{\nu>\beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\
&= \sum_{\nu\ge 1} \frac{\beta+1}{-\nu(-\nu-\beta)} + \sum_{\nu>\beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\
&= \sum_{\nu\ge 1} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\nu>\beta+\epsilon} \frac{\beta+1}{\nu(\nu-\beta)} \\
&\le \left[\sum_{1\le\nu\le\beta+1} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\beta+\epsilon<\nu\le 2\beta} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\nu>2\beta} \frac{\beta+1}{\nu(\nu+\beta)} \right] \\
&+ \left[\sum_{\beta+\epsilon<\nu\le 2\beta} \frac{\beta+1}{\nu(\nu-\beta)} + \sum_{\nu>2\beta} \frac{\beta+1}{\nu(\nu-\beta)} \right] \\
&\le \sum_{1\le\nu\le\beta+1} \frac{\beta+1}{\nu(\nu+\beta)} + \sum_{\beta+\epsilon<\nu\le 2\beta} \left[\frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \right] + \sum_{\nu>2\beta} \left[\frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \right].
\end{aligned}
$$

The first term is

$$
\le \sum_{1\le\nu\le\beta+1} \frac{\beta+1}{\nu(1+\beta)} = \sum_{1\le\nu\le\beta+1} \frac{1}{\nu} \ll \log(\beta+2).
$$

The second term is

$$
\begin{aligned}
&\sum_{\beta+\epsilon<\nu\le 2\beta} \frac{\beta+1}{\nu(\nu+\beta)} + \frac{\beta+1}{\nu(\nu-\beta)} \\
&\le \sum_{\beta+\epsilon<\nu\le 2\beta} \frac{\beta+1}{(\beta+1)(2\beta+1)} + \frac{\beta+1}{(\beta+1)(\nu-\beta)} \\
&= \sum_{\beta+\epsilon<\nu\le 2\beta} \frac{1}{2\beta+1} + \frac{1}{\nu-\beta} \\
&\ll 1 + \sum_{\beta+\epsilon<\nu\le 2\beta} \frac{1}{\nu-\beta} \ll 1 + \frac{1}{\epsilon} + \log(\beta+1) \ll_\epsilon \log(\beta+2).
\end{aligned}
$$

The third term is

$$\begin{aligned}
&= (\beta+1)\sum_{\nu>2\beta}\frac{2\nu}{\nu(\nu^2-\beta^2)} \\
&\le (\beta+1)\sum_{\nu>2\beta}\frac{2}{\nu^2-(\frac{\nu}{2})^2} \\
&\ll (\beta+1)\sum_{\nu>2\beta}\frac{1}{\nu^2} \ll 1.
\end{aligned}$$

Combing all these estimates, we find that the error term is

$$\ll_\epsilon \log(\beta+2).$$

This completes the proof. □

Theorem 2.6(van der Corput): Let $f \in C^2[a,b]$ be such that

$$|f''(t)| \approx \lambda > 0 \qquad (a < t < b).$$

Then we have

$$\sum_{a<n\le b} e(f(n)) \ll (b-a+1)\lambda^{1/2} + \lambda^{-1/2}. \tag{17}$$

Proof. If $\lambda > 1$, then (17) is satisfied trivially since we would have

$$\begin{aligned}
|\sum_{a<n\le b} e(f(n))| &\le \sum_{a<n\le b} |e(f(n))| \\
&\le b-a+1 \\
&\ll (b-a+1)\lambda^{1/2} + \lambda^{-1/2}.
\end{aligned}$$

Thus, we can assume that $\lambda \le 1$.

Let $\alpha := \inf_{a<t<b} f'(t)$, $\beta := \sup_{a<t<b} f'(t)$. By Theorem 2.5,

$$\begin{aligned}
\sum_{a<n\le b} e(f(n)) &= \sum_{\alpha-1/4<\nu<\beta+1/4}\int_a^b e(f(t)-\nu t)dt + O(\log(\beta-\alpha+2)) \\
&\le (\beta-\alpha+1)\max_{\alpha+\epsilon<\nu<\beta+\epsilon}\left|\int_a^b e(f(t)-\nu t)dt\right| + O(\log(\beta-\alpha+2).
\end{aligned} \tag{18}$$

Since $|f''(t)| \approx \lambda > 0$, there exists some positive constant c such that $\inf_{a<t<b}|f''(t)| = c\lambda$.

Furthermore, $f''(t)$ has constant sign on $[a, b]$. Now

$$|\frac{d^2}{dt^2}(f(t) - \nu t)| = |f''(t)|.$$

Applying Theorem 2.3, we have

$$|\int_a^b e(f(t) - \nu t)dt| \leq 4\sqrt{2/\pi c\lambda} \ll \lambda^{-1/2}.$$

The upper bound in (18) is thus

$$\ll (\beta - \alpha + 1)\lambda^{-1/2} + \log(\beta - \alpha + 2).$$

Note that

$$\beta - \alpha = \left| \int_a^b f''(t)dt \right| \ll \lambda(b - a).$$

Thus, the previous bound is

$$\begin{aligned}
&\ll (b - a)\lambda^{1/2} + \lambda^{-1/2} + \log(\lambda(b - a) + 2) \\
&\ll (b - a)\lambda^{1/2} + \lambda^{-1/2} + 1 + \lambda(b - a) \\
&\ll (b - a)\lambda^{1/2} + \lambda^{-1/2}.
\end{aligned}$$

□

We next establish a variant of Theorem 2.6 for a function of class C^3.

Theorem 2.7: Let $f \in C^3[a, b]$, with $b - a \geq 1$. Suppose that

$$|f'''(t)| \approx \lambda > 0 \qquad (a < t < b).$$

Then

$$\sum_{a<n\leq b} e(f(n)) \ll (b - a)\lambda^{1/6} + (b - a)^{1/2}\lambda^{-1/6} \tag{19}$$

To prove Theorem 2.7, we first prove the following lemma.

Lemma 2.8: Let f be a real valued function defined on $[a, b]$. For any integer q with $1 \leq q \leq b - a$, we have

$$\left| \sum_{a<n\leq b} e(f(n)) \right| \leq \frac{2(b - a)}{\sqrt{q}} + 2\left\{ \frac{(b - a)}{q} \sum_{r=1}^{q-1} \left| \sum_{a<n\leq b-r} e(f(n + r) - f(n)) \right| \right\}^{1/2}.$$

Proof. Let

$$F(t) = \begin{cases} e(f(t)) & \text{if } a < t \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Set $S := \sum_{n \in \mathbb{Z}} F(n)$ be the sum to be estimated. For any fixed m,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} F(n+m) &= \sum_{a < n+m \le b} e(f(n+m)) \\ &= \sum_{a < n' \le b} e(f(n')) \\ &= \sum_{n \in \mathbb{Z}} F(n). \end{aligned}$$

Thus, we have

$$S = \frac{1}{q} \sum_{m=1}^{q} \sum_{n \in \mathbb{Z}} F(n+m).$$

Interchanging the order of summation, we obtain

$$S = \frac{1}{q} \sum_{n \in \mathbb{Z}} \sum_{m=1}^{q} F(n+m).$$

Recall that the Cauchy-Schwarz Inequality says that for all complex numbers $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_n$, we have

$$\left| \sum_{i=1}^{n} z_i \overline{w_i} \right|^2 \le \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2, \tag{20}$$

where $\overline{w_i}$ denotes the complex conjugate of w_i. Applying this to S, we have

$$\begin{aligned} |S|^2 &= \frac{1}{q^2} \left(\left| \sum_{n \in \mathbb{Z}} \sum_{m=1}^{q} F(n+m) \right| \right)^2 \\ &\le \frac{1}{q^2} \sum_{n \in \mathbb{Z}}{}' 1 \cdot \sum_{n \in \mathbb{Z}}{}' \left| \sum_{m=1}^{q} F(n+m) \right|^2 \qquad (21) \\ &= \frac{1}{q^2} \sum_{n \in \mathbb{Z}}{}' 1 \cdot \sum_{n \in \mathbb{Z}}{}' \sum_{m,\, m'=1}^{q} F(n+m) \overline{F(n+m')}, \end{aligned}$$

where $'$ indicates that the summation is restricted to integers n with $a < n+m \le b$ for at least one m such that $1 \le m \le q$. Thus, $\sum_{n \in \mathbb{Z}}{}' 1$ does not exceed $b - a + q \le 2(b-a)$.

We can rewrite the inner sum and obtain

$$\begin{aligned}
&\sum_{m,\ m'=1}^{q} F(n+m)\overline{F(n+m')} \\
&= \sum_{m=1}^{q} F(n+m)\overline{F(n+m)} + \sum_{1\le m'<m\le q} \Big(F(n+m)\overline{F(n+m')} + F(n+m')\overline{F(n+m)} \Big)
\end{aligned} \tag{22}$$

Since

$$F(n+m')\overline{F(n+m)} = \overline{F(n+m)\overline{F(n+m')}},$$

the expression in (22) is equal to

$$q + 2\Re\left(\sum_{1\le m'<m\le q} F(n+m)\overline{F(n+m')} \right).$$

Since $\Re(\alpha) \le |\alpha|$, the second sum in (21), namely, $\sum_{n\in\mathbb{Z}}{}' \sum_{m,\ m'=1}^{q} F(n+m)\overline{F(n+m')}$, is at most

$$2(b-a)q + 2\Bigg| \sum_{1\le m'<m\le q} \sum_{n\in\mathbb{Z}} F(n+m)\overline{F(n+m')} \Bigg|. \tag{23}$$

Let $m+n=\nu$, $m-m'=r$, then ν runs through $\mathbb{Z}$ and $r\in\{1,2,\ldots,q-1\}$. If we let ν, r be fixed, then there are exactly $q-r$ solutions for $\{n,m,m'\}$, namely, $m'=j, m=j+r$, and $n=\nu-j-r$, where $1\le j\le q-r$. After performing a change of variables, we obtain that the expression in (23) is

$$\begin{aligned}
&\le 2(b-a)q + 2\Bigg| \sum_{r=1}^{q-1} (q-r) \sum_{\nu\in\mathbb{Z}} F(\nu)\overline{F(\nu-r)} \Bigg| \\
&\le 2q\Bigg\{ (b-a) + \sum_{r=1}^{q-1} \Bigg| \sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)} \Bigg| \Bigg\}
\end{aligned}$$

Inserting this upper bound into (21), we have

$$\begin{aligned}
|S|^2 &\le \frac{1}{q^2}\cdot 2(b-a)\cdot 2q\Bigg\{ (b-a) + \sum_{r=1}^{q-1} \Bigg| \sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)} \Bigg| \Bigg\} \\
&= \frac{4(b-a)^2}{q} + \frac{4(b-a)}{q} \sum_{r=1}^{q-1} \Bigg| \sum_{\nu\in\mathbb{Z}} F(\nu+r)\overline{F(\nu)} \Bigg| \qquad (24) \\
&= \frac{4(b-a)^2}{q} + \frac{4(b-a)}{q} \sum_{r=1}^{q-1} \Bigg| \sum_{a<n\le b-r} e(f(n+r))e(-f(n)) \Bigg|.
\end{aligned}$$

Since for all $a, b \geq 0$, we have $2\sqrt{ab} \geq 0$, we see that

$$a + 2\sqrt{ab} + b \geq a + b,$$

which indicates

$$(\sqrt{a} + \sqrt{b})^2 \geq (\sqrt{a+b})^2.$$

Thus,

$$\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}.$$

Applying this to (24), we find that

$$\begin{aligned} |S| &\leq \sqrt{\frac{4(b-a)^2}{q} + \frac{4(b-a)}{q} \sum_{r=1}^{q-1} \left| \sum_{a<n\leq b-r} e(f(n+r) - f(n)) \right|} \\ &\leq \frac{2(b-a)}{\sqrt{q}} + 2\left\{ \frac{(b-a)}{q} \sum_{r=1}^{q-1} \left| \sum_{a<n\leq b-r} e(f(n+r) - f(n)) \right| \right\}^{1/2}. \end{aligned}$$

□

Proof of Theorem 2.7. Let $g(x) := f(x+r) - f(x)$. Then for $x \in (a, b-r)$ with $1 \leq r < b-a$,

$$g''(x) = f''(x+r) - f''(x).$$

Since f'' is differentiable, Taylor's theorem of order 1 states

$$f''(x+r) = f''(x) + f'''(\xi)r$$

for some $x < \xi < x + r$. Thus,

$$|g''(x)| = |f'''(\xi)|r.$$

Since $|f'''(\xi)| \approx \lambda > 0$, we have $|g''(x)| \approx r\lambda$.
Let $L := b - a$. Applying Lemma 2.8, we have

$$\begin{aligned} \sum_{a<n\leq b} e(f(n)) &\ll Lq^{-1/2} + \left\{ Lq^{-1} \left| \sum_{a<n\leq b-r} e(f(n+r) - f(n)) \right| \right\}^{1/2} \\ &= Lq^{-1/2} + \left\{ Lq^{-1} \left| \sum_{a<n\leq b-r} e(g(n)) \right| \right\}^{1/2}. \end{aligned} \tag{25}$$

We now apply Theorem 2.6 to the $\sum_{a<n\le b-r} e(g(n))$ to see that this expression is

$$\begin{aligned}
&\ll Lq^{-1/2} + \left\{ Lq^{-1} \sum_{r=1}^{q-1} \left(L(r\lambda)^{1/2} + (r\lambda)^{-1/2} \right) \right\}^{1/2} \\
&\ll Lq^{-1/2} + \left\{ Lq^{-1} q \left(L(q\lambda)^{1/2} + (q\lambda)^{-1/2} \right) \right\}^{1/2} \\
&= Lq^{-1/2} + \left\{ L^2(q\lambda)^{1/2} + L(q\lambda)^{-1/2} \right\}^{1/2} \\
&\le Lq^{-1/2} + L(q\lambda)^{1/4} + L^{1/2}(q\lambda)^{-1/4}.
\end{aligned}$$

If λ satisfies $1 \le \lambda^{-1/3} \le L$, we can choose $q = \lfloor \lambda^{-1/3} \rfloor$. This makes the upper bound

$$\begin{aligned}
&\ll L\lambda^{1/6} + L\lambda^{1/6} + L^{1/2}\lambda^{-1/6} \\
&\ll (b-a)\lambda^{1/6} + (b-a))^{1/2}\lambda^{-1/6},
\end{aligned}$$

which is the desired result. The estimate (19) is trivially valid when $\lambda > 1$ or $\lambda < L^{-3}$. $\square$

Now we show the error term obtained using van der Corput's method in the Dirichlet divisor problem is $O_\epsilon(x^{1/3+\epsilon})$.

Theorem 2.8 (Voronoi, 1903). For $x \ge 2$, we have

$$\sum_{n\le x} d(n) = x(\log x + 2\gamma - 1) + O(x^{1/3}\log x). \tag{26}$$

Proof. The hyperbola method (Theorem 1.1) shows that the left-hand side of (26) equals

$$2\sum_{n=1}^{\lfloor\sqrt{x}\rfloor} \lfloor \frac{x}{n} \rfloor - \lfloor \sqrt{x} \rfloor^2. \tag{27}$$

Write $N := \lfloor \sqrt{x} \rfloor$ and let $B_1(t) = \{t\} - \frac{1}{2}$ denote the first Bernoulli function. Then (27) becomes

$$\begin{aligned}
2\sum_{n\le N} \lfloor \frac{x}{n} \rfloor - N^2 &= 2\sum_{n\le N} \left(\lfloor \frac{x}{n} \rfloor - \{\frac{x}{n}\} \right) - N^2 \\
&= 2\sum_{n\le N} \left(\frac{x}{n} - B_1\left(\frac{x}{n}\right) \right) - \frac{1}{2}\cdot 2\sum_{n=1}^{N} 1 - N^2 \\
&= 2\sum_{n\le N} \left(\frac{x}{n} - B_1\left(\frac{x}{n}\right) \right) - N - N^2.
\end{aligned}$$

Using Theorem 1.2 to estimate $\sum_{n\le N}\frac{1}{n}$, we now see that

$$\begin{aligned}\sum_{n\le x} d(n) =&\Big(2x\Big(\log N+\gamma+\frac{1}{2N}+O(\frac{1}{N^2})\Big)-N-N^2\Big)-2\sum_{n\le N}B_1(\frac{x}{n})\\ =&P(x)-2R(x),\end{aligned} \tag{28}$$

where

$$P(x):=2x\Big(\log N+\gamma+\frac{1}{2N}+O(\frac{1}{N^2}\Big)-N-N^2$$

and

$$R(x):=\sum_{n\le N}B_1(\frac{x}{n}).$$

Writing $N=\sqrt{x}-\theta$, with $0\le\theta<1$, we have

$$P(x)=2x\Bigg(\log(\sqrt{x}-\theta)+\gamma+\frac{1}{2(\sqrt{x}-\theta)}+O\Big(\frac{1}{x}\Big)\Bigg)-(\sqrt{x}-\theta)-(\sqrt{x}-\theta)^2.$$

Now

$$\log(\sqrt{x}-\theta)=\tfrac{1}{2}\log x+\log(1-\frac{\theta}{\sqrt{x}})=\tfrac{1}{2}\log x-\frac{\theta}{\sqrt{x}}+O(x^{-1}).$$

Furthermore,

$$\frac{1}{\sqrt{x}-\theta}=\frac{1}{\sqrt{x}(1-\theta/\sqrt{x})}=\frac{1}{\sqrt{x}}+O(x^{-1}).$$

Thus,

$$\begin{aligned}P(x)&=2x\Bigg(\tfrac{1}{2}\log x-\frac{\theta}{\sqrt{x}}+\gamma+\frac{1}{2\sqrt{x}}+O\Big(\frac{1}{x}\Big)\Bigg)-\sqrt{x}-x+2\theta\sqrt{x}+O(1)\\ &=(x\log x-2\theta\sqrt{x}+2\gamma x+\sqrt{x}+O(1))-\sqrt{x}-x+2\theta\sqrt{x}+O(1)\\ &=x\log x+(2\gamma-1)x+O(1).\end{aligned}$$

We now see that to obtain the desired result, it suffices for us to prove

$$R(x)\ll x^{1/3}\log x. \tag{29}$$

To establish (29), we apply van der Corput's technique. To begin with, note that $B_1(t)$ in our expression for $R(x)$ has the Fourier series

$$-\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\sin(2\pi jt)}{j},$$

which is not absolutely convergent. If it were, we could work directly with it. Since it is not, we avoid technical difficulties by introducing the related function

$$B(t) := \frac{1}{2}J\int_{-\frac{1}{J}}^{\frac{1}{J}} B_1(t+u)du$$

for J large. It is not difficult to show that $B(t)$ is Lipschitz of order 1, so its Fourier series is absolutely convergent. To compute it, one calculates

$$\begin{aligned} A_j = \int_0^1 B(t)e(-jt)dt =& \frac{J}{2}\int_{-\frac{1}{J}}^{\frac{1}{J}} \left(\int_0^1 B_1(t+u)e(-jt)dt\right)du \\ & \frac{J}{2}\int_{-\frac{1}{J}}^{\frac{1}{J}} e(ju)\left(\int_0^1 B_1(y)e(-jy)dy\right)du \\ & -\frac{J}{2}\int_{-\frac{1}{J}}^{\frac{1}{J}} \frac{e(ju)}{2\pi ij}du = \frac{J}{2\pi^2 j^2}\frac{\sin(2\pi j/J)}{2i}. \end{aligned} \tag{30}$$

Hence,

$$\begin{aligned} B(t) &= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} A_j e(jt) = \sum_{j=1}^{\infty} A_j\big(e(jt) - e(-jt)\big) \\ &= \sum_{j=1}^{\infty} a_j \sin(2\pi jt), \end{aligned} \tag{31}$$

where

$$a_j = \frac{J}{2\pi^2 j^2}\sin(2\pi j/J).$$

Next let

$$h(t) := |B(t) - B_1(t)|.$$

Then $h(t)$ is also Lipschitz of order 1. In fact,

$$h(t) = \frac{1}{2}(1 - J\|t\|)^+ \qquad (J > 1).$$

Calculating the Fourier series of $h(t)$, one obtains

$$h(t) = \frac{1}{2J} + \sum_{j=1}^{\infty} b_j \cos(2\pi jt), \tag{32}$$

where

$$b_j = \frac{J}{\pi^2 j^2} \sin^2(\frac{\pi j}{J}).$$

Since $|\sin x| \leq \min(1, x)$, we see that

$$|a_j| \ll \min(J/j^2, 1/j).$$

Similarly,

$$|b_j| \ll \min(J/j^2, 1/j).$$

Thus,

$$|a_j| + |b_j| \ll \min(j, J)/j^2 \qquad (j \geq 1). \tag{33}$$

We are now ready to estimate $R(x)$. For $M < T \leq 2M$, we set

$$R(x; M, T) := \sum_{M<n\leq T} B_1(\frac{x}{n}). \tag{34}$$

By our definition of $h(t)$, we have

$$|R(x; M, T) - \sum_{M<n\leq T} B(\frac{x}{n})| \leq \sum_{M<n\leq T} h(\frac{x}{n}).$$

By (31) and (32), we have

$$\begin{aligned} R(x; M, T) &= \sum_{M<n\leq T} B(\frac{x}{n}) + O\Big(\sum_{M<n\leq T} h(\frac{x}{n})\Big) \\ &= \sum_{M<n\leq T} \sum_{j=1}^{\infty} a_j \sin(2\pi jx) + O\Big(\sum_{M<n\leq T} \Big(\frac{1}{2J} + \sum_{j=1}^{\infty} b_j \cos(2\pi jx)\Big)\Big) \\ &= \sum_{M<n\leq T} a_j \sum_{j=1}^{\infty} \sin(2\pi jx) + O\Big(\frac{M}{J} + \sum_{j=1}^{\infty} b_j \Big| \sum_{M<n\leq T} \cos(2\pi jx)\Big|\Big). \end{aligned}$$

For each real number y, let $f(u) = y/u$. Then,

$$f''(n) = \frac{2y}{n^3} \asymp \frac{y}{M^3} \qquad (M < n < T \leq 2M).$$

By Theorem 2.6,

$$\begin{aligned}\sum_{M<n\le T} e\Big(\frac{y}{n}\Big) &\ll (2M-M+1)(\frac{y}{M^3})^{1/2} + (\frac{y}{M^3})^{-1/2} \\ &\ll \Big(\frac{y}{M}\Big)^{1/2} + \Big(\frac{M^3}{y}\Big)^{1/2}.\end{aligned} \tag{35}$$

Applying this estimate with $y = jx$,

$$\sum_{M<n\le T} e\Big(\frac{jx}{n}\Big) \ll (\frac{jx}{M})^{1/2} + (\frac{M^3}{jx})^{1/2}.$$

Thus, from our bound (33) we have,

$$\begin{aligned}R(x;M,T) &\ll \frac{M}{J} + \sum_{j=1}^{\infty}(|a_j|+|b_j|)\Bigg(\Big(\frac{jx}{M}\Big)^{1/2} + \Big(\frac{M^3}{jx}\Big)^{1/2}\Bigg) \\ &\ll \frac{M}{J} + \sum_{j=1}^{\infty}\frac{\min(j,J)}{j^2}\Bigg(\Big(\frac{jx}{M}\Big)^{1/2} + \Big(\frac{M^3}{jx}\Big)^{1/2}\Bigg) \\ &\ll \frac{M}{J} + \sum_{j=1}^{J}\frac{j}{j^2}(\frac{jx}{M})^{1/2} + \sum_{j=J}^{\infty}\frac{J}{j^2}(\frac{jx}{M})^{1/2} + \sum_{j=1}^{\infty}(\frac{M^3}{j^3x})^{1/2} \\ &\ll \frac{M}{J} + \sum_{j=1}^{J}\frac{1}{j^{1/2}}(\frac{x}{M})^{1/2} + \sum_{j=J}^{\infty}\frac{J}{j^{3/2}}(\frac{x}{M})^{1/2} + \sum_{j=1}^{\infty}(\frac{M^3}{x})^{1/2}\frac{1}{j^{3/2}} \\ &\ll \frac{M}{J} + J^{1/2}(\frac{x}{M})^{1/2} + \sqrt{J}(\frac{x}{M})^{1/2} + (\frac{M^3}{x})^{1/2} \\ &\ll \frac{M}{J} + \Big(\frac{Jx}{M}\Big)^{1/2} + \Big(\frac{M^3}{x}\Big)^{1/2}.\end{aligned}$$

To optimize this bound, we set $J := Mx^{-1/3}$ and obtain

$$\begin{aligned}R(x;M,T) &\ll \frac{M}{Mx^{-1/3}} + (\frac{Mx^{2/3}}{M})^{1/2} + (\frac{M^3}{x})^{1/2} \\ &\ll x^{1/3} + \Big(\frac{M^3}{x}\Big)^{1/2}.\end{aligned}$$

Since we must have $J > 1$, this bound holds provided $M > x^{1/3}$. However, it also holds when $M \le x^{1/3}$ trivially because by (34) we have $R(x,M,T) \ll M \ll x^{1/3}$.

To obtain our bound for $R(x)$ we use our bound for $R(x,M,T)$ over dyadic intervals. Set

$r_0 := \lfloor \frac{\log N}{\log 2} \rfloor - 1$. Then, we have

$$\begin{aligned} R(x) &= \sum_{r=-1}^{r_o} R(x; 2^r, 2^{r+1}) + R(x; 2^{r_o+1}, N) \\ &\ll \sum_{r=-1}^{r_o} \left(x^{1/3} + \left(\frac{2^{3r}}{x} \right)^{1/2} \right) + x^{1/3} \\ &\ll r_0 x^{1/3} + N^{3/2} x^{-1/2}. \end{aligned}$$

Recall, $N = \lfloor \sqrt{x} \rfloor$. Thus, we have

$$\begin{aligned} R(x) &\ll (\frac{\log \sqrt{x}}{\log 2} - 1) x^{1/3} + x^{3/4} x^{-1/2} \\ &\ll x^{1/3} \log x, \end{aligned}$$

which is our stated result in (29). Thus, we can conclude that

$$\sum_{n \leq x} d(n) = P(x) - 2R(x) = x(\log x + 2\gamma - 1) + O(x^{1/3} \log x).$$

This completes the proof. □

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6 Reference

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