Differential Topology: A Survey

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1 Preliminaries

Differential topology serves in many ways as a bridge between many of the broad, global results of general topology and the local, analytical results of differential geometry. As we see in the foregoing work, fundamental quantities known to topology such as the Euler characteristic, singular homology, and the like, may be expressed in terms of fairly simple analysis. Work in this area was pioneered by notable mathematicians such as Poincaré, Hopf, and Lefschetz, but much of our discussion will focus on contributions made by Marston Morse. Originally, Morse studied the general theory of calculus of variations, an area of mathematics quite commonly viewed only in relation to its physical applications in physics. However, although this topic seems removed from pure mathematics to the uninitiated, Morse's work ultimately showed that critical point theory, and hence differential topology, are at the heart of it all.

2 Background

To begin, we recall the extrinsic definition of a smooth manifold as a subset of Euclidean space as well as some of the rudimentary associated results. The most fundamental object in our study is the manifold, which we will define shortly. However, to give a precise notion of smoothness we first state

Definition 1 For open sets $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$, a map $f : U \to V$ is smooth if every partial derivative $\frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}}$ exist and are continuous. More generally, if $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are arbitrary subsets, then $f : X \to Y$ is called smooth if for $x \in X$, $\exists U \subset \mathbb{R}^k$ open with $x \in U$ and a smooth map $F : U \to \mathbb{R}^l$ such that $F\big|_{U \cap X} = f$.

Having now thoroughly specified what we mean by differentiability in this context, we are able to formally define the most basic object.

Definition 2 $M \subset \mathbb{R}^k$ is an (smooth) m-manifold if each $x \in M$ has a neighborhood $U \cap M$ such that $U \cap M \approx_D V$ (diffeomorphic), V being some open subset of $\mathbb{R}^m$. A diffeomorphism $f : V \to U \cap M$ is a *parametrization*, while the inverse $f^{-1} : U \cap M \to V$ is a *local coordinate system.*

The most natural way to continue developing our theory is to study maps $f : M \to N$ between manifolds. Along the way we will construct the tangent space of a manifold at a point, but for now consider it as a linear subspace T_xM, $x \in M$, of the ambient $\mathbb{R}^k$. Much like the derivative of a function $g : \mathbb{R} \to \mathbb{R}$ produces a tangent line, the same is true for more general manifolds where to the map $f : M \to N$ there is an associated linear map $df_x : T_xM \to T_{f(x)}N$ for every $x \in M$. Properties (1) - (3) below give a somewhat clearer picture of this linear map.

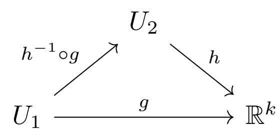
(1) For $f : U \to V$, $g : V \to W$ smooth maps, $d(g \circ f)_x = dg_{f(x)} \circ df_x$ (can phrase in terms of commutative triangles).
(2) If $I_U : U \to U$ is the identity map of an open set $U \subset \mathbb{R}^k$, then $dI_x : T_xU = \mathbb{R}^k \to T_xU = \mathbb{R}^k$ is the identity map of $\mathbb{R}^k$.
(3) If $L : \mathbb{R}^k \to \mathbb{R}^l$ is a linear mapping, $dL_x = L$.

Taking these properties as given, we prove a useful proposition:

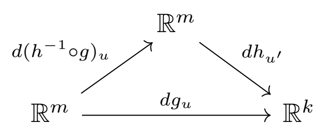
Proposition 1 Given a diffeomorphism $f : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^l$, $\implies$ $k = l$ and $df_x : \mathbb{R}^k \to \mathbb{R}^l$ must be nonsingular.

Proof. Because f is a diffeomorphism, f^{-1} exists and is smooth, so $f^{-1} \circ f = Id_U$, therefore by (1) and (2) we know $d(f^{-1} \circ f) = d(f^{-1}) \circ df = Id_{\mathbb{R}^k}$, and similarly $df \circ d(f^{-1}) = Id_{\mathbb{R}^l}$. This implies df has a two-sided inverse, i.e. it is nonsingular, and then we must have $k = l$. $\square$

Now for the formal definition of a tangent space that was promised.
First, make a choice of parametrization $g : U \subset \mathbb{R}^m \to M \subset \mathbb{R}^k$ be a parametrization of the neighborhood $V = g(U)$ of $x \in M$. For a specific $u \in U$, let $g(u) = x$; as a map from U to $\mathbb{R}^k$, it has derivative $dg_u : \mathbb{R}^m \to \mathbb{R}^k$. Then, set $T_xM = dg_u(\mathbb{R}^m) \subset \mathbb{R}^k$. To be fully rigorous, we need to verify that this definition is independent of the choice of g. To do so, suppose we have another parametrization $h : U' \subset \mathbb{R}^m \to M \subset \mathbb{R}^k$ of a neighborhood $V' = h(U')$ of $x \in M$, setting $h^{-1}(x) = u'$. Then we know $h^{-1} \circ g$ is a diffeomorphism between some neighborhood U_1 of u and a neighborhood U_2 of u'. From this we can construct a commutative diagram of diffeomorphisms



and in turn a commutative diagram of linear transformations



Because $h^{-1} \circ g$ is a diffeomorphism, its derivative is a linear isomorphism, and hence it is clear that we have

Definition 3 $T_xM = Im(dg_u) = Im(dh_{u'})$.

We have reached the point where our focus may narrow somewhat. To that end we wish to identify useful aspects of the linear map df_x or simply df. As this falls within the well known area of linear algebra, it is reasonable to expect pleasant results.

Definition 4 For a smooth map $f : M \to N$ between manifolds which (for now) are assumed to be of equal dimension, $x \in M$ is called a regular point of f if the linear map df_x is nonsingular, whereas x is called a critical point if df_x is singular. The image of x under f, $f(x) \in N$, is a regular (respectively critical) value.

Proposition 2 If M is compact and $y \in N$ is a regular value, it follows that $|f^{-1}(y)|$ is finite.

Proof. Suppose $f^{-1}(y)$ is an infinite set; then we can form a sequence $(x_n)_{n=1}^{\infty} \subseteq f^{-1}(y)$ which by compactness of M must have a convergent subsequence, $(x_{n_k})_{k=1}^{\infty}$, $x_{n_k} \to x$. By the topological definition of convergence, this means every open neighborhood of x must contain some element x_{n_k} of the convergent subsequence. But, this contradicts the fact that f must map an open neighborhood of x diffeomorphically to an open neighborhood of y (the map cannot be 1-1, as $f(x) = f(x_{n_k}) = y$ for $x \neq x_{n_k}$.) $\square$

In a similar vein, with the same restrictions on M, we have the following:

Proposition 3 As a function of y, $|f^{-1}(y)|$ is locally constant. That is, $\exists$ a neighborhood V of y such that for any $y' \in V$ regular value, $|f^{-1}(y)| = |f^{-1}(y')|$.

Proof. Let $f^{-1}(y) = \{x_1, \ldots, x_k\}$; choose neighborhoods U_i of each x_i which are pairwise disjoint (possible because in the setup where M is compact, $f^{-1}(y)$ is finite and discrete) and map diffeomorphically to neighborhoods V_i of y; define $V = \cap V_i - f(M - \cup U_i)$. $\square$

The preceding propositions give the first indication as to where differential topology leads, as we will see $|f^{-1}(y)|$ play a central role. To conclude this section, we first state Sard's Theorem which is of general importance, but perhaps more interestingly we will prove what is commonly known as the Regular Value Theorem.

Theorem 1 (Sard) Let $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be smooth, U an open set. Denote by C the set of critical points of f, i.e. $C = \{x \in \mathbb{R}^m \,|\, rank(df_x) < n\}$. Then $f(C) \subset \mathbb{R}^n$ has Lebesque measure zero. [Interesting cases only when $m \geq n$, as $m < n \implies C = U$.]

To move to the more general setting of manifolds, we need only recall that a smooth manifold M is coverable by a countable collection of open sets, each of which is diffeomorphic to an open set $U \subseteq \mathbb{R}^m$. We immediately obtain

Corollary 1.1 The set of regular values of a smooth map $f : M \to N$ is everywhere dense in N.

Theorem 2 Let $f : M \to N$ be a smooth map between manifolds of dimensions m and n respectively, with $m \geq n$. If $y \in N$ is a regular value of f, then we have $f^{-1}(y)$ is a smooth submanifold of M of codimension n.

Proof. (Regular Value) Let $x \in f^{-1}(y)$; then by definition of a regular value, df_x surjectively maps the m-dimensional vector space T_xM to the n-dimensional vector space T_yN. From the rank-nullity theorem of linear algebra, we know that the null space R of df_x must have dimension $m - n$. We may assume that $M \subseteq \mathbb{R}^K$ for some K sufficiently large, and then define a linear transformation $L : \mathbb{R}^K \to \mathbb{R}^{m-n}$ that is nonsingular on R. Then, construct $F : M \to N \times \mathbb{R}^{m-n}$ by $F(z) = (f(z), L(z)) \implies dF = (df, L)$ which is nonsingular. Therefore by the IFT, F must map an open neighborhood of x diffeomorphically to a neighborhood of $(y, L(x))$. Also note under F that $f^{-1}(y)$ corresponds to the hyperplane $\{y\} \times \mathbb{R}^{m-n}$ (since $F(f^{-1}(y))$ is this by definition). Indeed then F is a diffeomorphism between $f^{-1}(y) \cap$ (open neighborhood of x) and $(y \times \mathbb{R}^{m-n}) \cap$ (open neighborhood V of $(y, L(x))$), completing the proof. □

The Regular Value Theorem is especially useful, yielding many examples of manifolds which we do not have to messily define via charts and diffeomorphisms. Consider as a specific example the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by $f(x_1, \ldots, x_{n+1}) = \sum x_i^2$. We see that the only critical point is where $\nabla f = (2x_1, \ldots, 2x_{n+1}) = \vec{0} \iff (x_1, \ldots, x_{n+1}) = \vec{0}$. It follows that the only critical value is $f(\vec{0}) = 0$, so $f^{-1}(r)$, $r \in \mathbb{R}\backslash\{0\}$, is a smooth submanifold of $\mathbb{R}^{n+1}$ of dimension $(n+1) - 1 = n$. Moreover, when $r = 1$ we see $f^{-1}(1) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, \sum x_i^2 = 1\}$, a set which is S^n. In this fashion we are able to identify the n-dimensional sphere as a smooth manifold without (for the moment) needing to deal with stereographic projection.

To proceed, we need to develop the notion of a normal space, at least minimally.

Definition 5 Let $M \subseteq N$ be a submanifold; we then know immediately that $T_xM \subseteq T_xN$ (linear subspace) for all $x \in M$. Then we can construct $(T_xM)^\perp = \{v \in T_xN \,|\, v \perp T_xM\} \subseteq T_xN$ called the normal space of M, and has dimension $\dim(M) - \dim(N)$.

Lemma 1 Let $f^{-1}(y) = M' \subseteq M$ be a submanifold, and $f : M \to N$ be a smooth map having $y \in N$ as a regular value. Then the null space R of $df_x : T_xM \to T_yN$ is precisely T_xM', which implies $(T_xM')^\perp \cong T_yN$ under df_x.

To make greater use of the power of the previous theorem, we need to broaden our usual view of a manifold to include objects with edges or boundaries. This is accomplished simply by replacing $\mathbb{R}^m$ in the definition of a smooth m-manifold with $\mathbb{H}^m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \,|\, x_1 \geq 0\}$. Now our theory can expand and include objects such as the cylinder $S^1 \times I$ where $I = [0, 1]$, and we reformulate the Regular Value Theorem to accomodate.

Theorem 3 Take a smooth map $f : X^m \to N^n$ with $\partial X \neq \emptyset$, $m > n$. If $y \in N$ is a regular value of f *and* of $\partial f = f\big|_{\partial X^m}$, it follows that $f^{-1}(y)$ is a smooth submanifold of X^m of codimension n, and $\partial f^{-1}(y) = f^{-1}(y) \cap \partial X^m$.

Proof. We restrict our attention to the case $M = \mathbb{H}^m$, $N = \mathbb{R}^n$, which can be extended naturally to more general manifolds with boundary via parametrizing diffeomorphisms. Let $x \in f^{-1}(y)$ for $y \in \mathbb{R}^n$ a regular value. If $x \in Int(\mathbb{H}^m)$, we are done as we can apply the Regular Value Theorem for boundaryless manifolds. Hence we can assume we are in the case of greater interest, $x \in \partial\mathbb{H}^m$. We can define a map $g : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ on a neighborhood of x in $\mathbb{R}^m$ such that $g\big|_{U \cap \mathbb{H}^m} = f$. We then have that $g^{-1}(y)$ is a submanifold of $\mathbb{R}^m$ of dimension $m - n$, restricting U small enough to avoid critical points of g as needed.

Claim: The projection $\pi_m : \mathbb{R}^m \to \mathbb{R}$ has 0 as a regular value.

To see this, note that for $x \in \pi_m^{-1}(0)$, $T_x(g^{-1}(y)$ is the null space of $df_x = df_x$. Our hypotheses stipulate that f has x as a regular point, as well as $f\big|_{\partial \mathbb{H}^m}$, so the null space is not completely contained in $\mathbb{R}^{m-1} \times \{0\}$. $\therefore g^{-1}(y) \cap \mathbb{H}^m = f^{-1}(y) \cap U$. □

3 Intersection Numbers and Degrees

Throughout this section, the following setup will be used: (i) M is compact; (ii) $\partial M = \emptyset$; (iii) N is connected; (iv) $\dim M = \dim N$. These hypotheses are fairly restrictive, but they allow us to get to several results at the heart of differential topology.

3.1 Modulo 2 Version

Definition 6 For $X \subset \mathbb{R}^k$ and maps $f, g : X \to Y$, f is smoothly homotopic to g, denoted $f \sim g$, if there is a smooth map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

Proposition 4 Smoothly homotopic is an equivalence relation.

Definition 7 If $f, g : X \to Y$ are diffeomorphisms, we may say f is smoothly isotopic to g if there is a smooth homotopy between them as above with the particular property that for each $t \in I$, $x \to F(x, t)$ is a diffeomorphism between X and Y.

Lemma 2 (Homotopy Invariance) Let $f, g : M^n \to N^n$, $f \sim g$. If $y \in N$ is a regular value of both f and g, then $|f^{-1}(y)| \equiv |g^{-1}(y)| \mod 2$.

Lemma 3 (Homogeneity) Let $y, z \in Int(N)$. Then there is a diffeomorphism $h : N \to N$ which satisfies (1) h is smoothly isotopic to Id_N, (2) $h(y) = z$.

Proof. An outline of the proof is: (a) Define an isotopy $\mathbb{R}^n \to \mathbb{R}^n$ fixing points outside the open unit ball, slides $\vec{0}$ to desired point in open unit ball; (b) The interior points of N have neighborhoods diffeomorphic to $\mathbb{R}^n$, hence (a) $\implies$ points sufficiently close to y can be mapped to it or vice-versa via an isotopy such as h; (c) Isotopy classes of N are open sets and partition it into a disjoint union of open sets. From the setup at the beginning of the section, N is connected, so there can only be one such class. □

Theorem 4 If y, z are regular values of $f : M \to N$, and $f \sim g$, then $|f^{-1}(y)| \equiv |g^{-1}(z)| \mod 2$.

Application: Consider $c : M \to M$, a constant map from the manifold M to itself. It is clear from the homotopy invariance of mod 2 degrees that such a map has degree 0 modulo 2. On the other hand we see the identity map $id : M \to M$ satisfies $|id^{-1}(y)| = 1 \mod 2$ for any $y \in M$. Therefore we may conclude that the identity map on a manifold which is compact and boundaryless cannot be null homotopic. For further specificity we may consider $M = S^n$, and by applying the result just mentioned we deduce that there is no smooth retraction map $f : D^{n+1} \to S^n$ (as supposing one exists leads to the existence of a smooth homotopy $F : S^n \times I \to S^n$ between the identity and a constant, a contradiction).

3.2 Brouwer Degree

We have now seen that degree theory modulo 2 yields a topological invariant. Next, we hope to construct a similar invariant which contains more information and is thus capable of more finely distinguishing between manifolds. This requires that we develop the concept of an orientation of a manifold, i.e. to make precise the fundamental difference between, say, the Möbius strip and cylinder.

Definition 8 An orientation for a finite dimensional vector space is an equivalence class of ordered bases: an ordered basis $(e_1, \dots, e_n)$ gives the same orientation as $(e'_1, \dots, e'_n)$ if $e'_i = \sum a_{ij} e_j$ with $\det(a_{ij}) > 0$. Their orientations are opposite if $\det(a_{ij}) < 0$.

Specific Example: To elucidate the functionality of the above definition, consider the standard basis in its usual order on $\mathbb{R}^n$: $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$. This gives the standard orientation; suppose instead we had the basis elements listed in the opposite order to give a new orientation $\{e'_1 = e_n, e'_2 = e_{n-1}, \ldots, e'_n = e_1\}$. Then notice $e'_{n+1-i} = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots 0 & \end{bmatrix} e_i$. If we denote this matrix by A, $\det A = (-1)^n$. For familiarity, consider the familiar $\mathbb{R}^3$. Then we see that by labeling $x = e_1$, $y = e_2$, and $z = e_3$, we recover the notion of a "right-handed" or "left-handed" coordinate system, each of which corresponds to a distinct orientation of the space.

Definition 9 An <u>oriented manifold</u> M is one with a consistent choice of orientation on each T_xM. By consistent we mean for each x in M there is a neighborhood U with $U \approx^D V \subseteq \mathbb{R}^m$ or $\mathbb{H}^m$; this diffeomorphism should be orientation-preserving in that for each $x \in U$, the isomorphism df_x should carry the orientation of T_xM to the standard orientation of $\mathbb{R}^m$.

Note that in this definition we have included the case of manifolds with boundary, but this seems nonintuitive; since the boundary of a manifold is a smooth manifold in its own right, albeit of one dimension less, is not the tangent space there of one smaller dimension as well? This would mean a reduction in the size of the basis, leading to inconsistencies. Instead, by convention the tangent space at a boundary point is of the same dimension but as three types of vectors: those in $T_x(\partial M)$, outward pointing vectors forming a half-open space bounded by $T_x(\partial M)$, and inward pointing vectors.

Definition 10 Let M, N be oriented manifolds both of dimension n without boundary. Given a map $f : M \to N$, its <u>(Brouwer) degree</u> is $deg(f; y) = \sum_{x \in f^{-1}(y)} sign(df_x)$, where $y \in N$ is a regular value.

Note that given the dependence on a regular value y, as in the case of deg_2, we see this is a locally constant function of y. This leads us to a slew of important theorems analogous to those proven under modulo 2 conditions.

Theorem 5 (A): $deg(f; y) \in \mathbb{Z}$ is independent of the choice of regular value y, hence can simply be denoted by $deg(f)$ (assuming N is connected).

Theorem 6 (B): If $f \sim g$, $\implies deg(f) = deg(g)$.

To prove these first two theorems, we build up our machinery with lemmas.

Lemma 4 Suppose X is a compact, oriented manifold with $\partial X = M$ oriented accordingly. If $f : M \to N$ extends to a smooth map $F : X \to N$ such that $F\big|_M = f$, then $deg(f; y) = 0$ for all regular values $y \in N$.

Proof. Suppose $y \in N$ is a regular value of F and f (possible by Sard). Then the regular value theorem implies $F^{-1}(y)$ is a smooth, compact 1-manifold, i.e. a finite collection of circles and closed line segments whose endpoints fall on $\partial X = M$. Let S =one such segment, $\partial S = \{a\} \cup \{b\}$. Our goal is to show that $sign(df_a) + sign(df_b) = 0$, as then summing over all such arcs we will find the degree to be 0 as desired. Note that the given orientations of X and N endow S with an orientation as well: for $x \in S$, let $(v_1, \ldots, v_{n+1})$ be a positively oriented basis for T_xX, so v_1 is tangent to S; then v_1 determines the required orientation of $T_xS \iff dF_x$ maps $(v_2, \ldots, v_{n+1})$ to a positively oriented basis of T_yN. If $v_1(x)$ is the aforementioned unit tangent vector to S at x, we know it is smooth as a function of x and moreover that it must then be inward pointing WLOG at a, and outward pointing at b. Thus $sign(a) = -1$, $sign(b) = 1$, giving the result.
Suppose alternatively that y is a regular value of f but not of F. We know though there is some open neighborhood of y such that $def(f; y)$ is constant on all of U, so from Sard's theorem we can find a common regular value y' shared by f and F. This resolves any issues and completes the proof. □

Lemma 5 Let $F : M \times I \to N$ be a smooth homotopy between $f, g : M \to N$. Then if y is a regular value of both f and g, $deg(g; y) = deg(f; y)$. Thus the integer degree, much like the $\mathbb{Z}/2$ degree yet more powerful, is a smooth homotopy invariant.

Proof. Orient $M \times I$ as a product; then $\partial(M \times I) = M \times \{0\} \cup M \times \{1\}$ where $M \times \{0\}$ will have the wrong orientation and $M \times \{1\}$ will have the correct orientation. Then by the previous Lemma, $deg(F\big|_{\partial(M\times I)}; y) = deg(g; y) - deg(f; y) = 0$, from which the result follows. $\square$

With a Lemma in either hand, the Theorems of this section are proven in analogous fashion to their modulo 2 versions.

Application: Consider the family of maps $f_k : \mathbb{C} \to \mathbb{C}$ given by $f_k(z) = z^k$ for $k \in \mathbb{Z}$. In particular, we can easily see $f_k\big|_{S^1} : S^1 \to S^1$ maps the 1-sphere to itself as a k-fold covering. This map behaves nicely (even if $k < 0$, $(0,0) \notin S^1$), and we can compute that $deg(f_k) = k$. Unlike in previous sections when we could only have said $def(f_k) = k \mod 2$, which would only differentiate between even and odd values of k, integer degree theory separates all of these maps into distinct homotopy classes.

Moving forward, we wish to prove a general version of what is colloquially known as the Hairy Ball Theorem. To that end we must first make a definition.

Definition 11 A tangent vector field on $M \subset \mathbb{R}^k$ is a smooth assignment of vector $v : M \to \mathbb{R}^k$ such that $v(x) \in T_xM$ for all $x \in M$.

As previously mentioned, we are interested in the case $M = S^n$. The definition is satisfied there $\iff$ $v(x) \cdot x = 0$ for all $x \in S^n$. Moreover, if we *assume* $v(x) \neq 0$ for all x, we may assume $v(x) \cdot v(x) = 1$ (otherwise may normalize without fear of division by zero). Consequently, we may treat v not as a map $S^n \to \mathbb{R}^{n+1}$ but as a map $S^n \to S^n$. Now, define a function $F : S^n \times [0, \pi] \to S^n$ given by $F(x, \theta) = x \cos\theta + v(x) \sin\theta$. We can compute that $F \cdot F = 1$ for any x, θ, $F(x, 0) = x$, and $F(x, \pi) = -x$. Therefore, F provides a homotopy between the antipodal map $A(x) = -x : S^n \to S^n$, and Id_{S^n}. However, we saw before that this is impossible for even n, hence we arrive at a contradiction in that case. This tells us that $\exists$ nonvanishing vector field on $S^n \implies n$ odd. To obtain the converse, let $n = 2k - 1$ be odd; then note that $v(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$ explicitly defines a nonvanishing vector field. We have at this point succeeded in proving

Theorem 7 A nonvanishing tangent vector field exists on $S^n \iff n$ is odd.

4 Vector Fields; Poincaré-Hopf Theorem

To approach the next topic, let $U \subseteq \mathbb{R}^m$ be open, and $v : U \to \mathbb{R}^m$ be a smooth vector field. Additionally, suppose v has an isolated zero at some $z \in U$. Then taking a sphere S_r of radius r (sufficiently small such that $S_r \subseteq U$) and dimension $m - 1$ centered at z, we may define $\hat{v} : S_r \to S^{m-1}$ by $\hat{v}(x) = \frac{v(x)}{\|v(x)\|}$. We refer to the degree of $\hat{v}$ as the index i of v at z.

Definition 12 Vector fields v, v' on M and N respectively correspond under f if df_x satisfies $v(x) \mapsto v'(f(x))$.

Definition 13 Let $g : U \subseteq \mathbb{R}^m \to M$ be a parametrization of a neighborhood V of a point $z \in M$.

Note that if f is a diffeomorphism, the corresponding vector field v' is uniquely determined by v. From here on we will define $v' := df \circ v \circ f^{-1}$ in such a situation.

Lemma 6 Suppose that v, a vector field on U, corresponds to $v' = df \circ v \circ f^{-1}$ on U' under a diffeomorphism $f : U \to U'$. Then the index of v at an isolated zero z is equal to the index of v' at $f(z)$.

Using Lemma 6, we can define the notion of an index of a zero of a vector field on a manifold.

Definition 14 If $g : U \subseteq \mathbb{R}^m \to M$ is a parametrization of a neighborhood of $z \in M$, the index of a vector field v on M having a zero at z is the corresponding index of $v' = dg^{-1} \circ v \circ g$ on U at the zero $g^{-1}(z)$.

Let M be a compact manifold with w a vector field on M having isolated zeros. If $\partial M \neq 0$, then the vector field must be *outward pointing* at all $x \in \partial M$.

Theorem 8 (Poincaré-Hopf) The sum of the indices, $\sum i$, at the zeros of w is equal to the Euler characteristic of M:

$$\sum i = \chi(M) = \sum (-1)^i rank H_i(M).$$

As a consequence, $\sum i$ is a topological invariant of the manifold M; it must therefore be independent of the choice of w.

This is a result of great theoretical importance and depth. The sum of the indices of a vector field is a concept rooted in analysis, while the Euler characteristic is a fundamental topological invariant, and so a link is forged between two very distinct areas of mathematics. This connection will be revisited when the Gauss-Bonnet Theorem is covered.

5 Framed Cobordism Classes and the Pontryagin Construction

If not previously mentioned, to be clear: the degree of a map $f : M \to N$ as it has been defined heretofore is defined only when $\dim M = \dim N$ and when both M and N are oriented. These are rather restrictive conditions, but Pontryagin saw a way to generalize. Given M, any compact, boundaryless manifold we may consider maps $f : M \to S^n$. Through this new construction we will see there is a one to one correspondence between the homotopy classes of maps and the framed cobordism classes of submanifolds, an especially useful relation in the study of homotopy of spheres.

Definition 15 Let N, N' be n-dimensional submanifolds of M, both boundaryless. We say N is cobordant to N' within M if $N \times [0,\epsilon) \cup N' \times (1-\epsilon, 1] \subset M \times [0,1]$ can be extended to a compact manifold $X \subseteq M \times [0,1]$ such that $\partial X = N \times \{0\} \cup N' \times \{1\}$ and $X \cap (M \times \{0\} \cup M \times \{1\}) = \partial X \cap (M \times \{0\} \cup M \times \{1\})$.

Definition 16 Let $N \subset M$ be a submanifold. A framing of N is a function $b : N \to (T_x N)^{\perp} \subset T_x M$, $b(x) = (v^1(x), \ldots, v^{m-n}(x))$ assigning a basis to the normal space at x. We call the pair (N, b) a framed submanifold.

Note that in the language of vector bundles which are commonly used the above notion of a framing can be stated as a trivialization of the normal bundle of N. From these two constructive definitions, we form a third.

Definition 17 Two framed submanifolds (N, b) and (N', b') are framed cobordant if there is a cobordism $X \subset M \times [0,1]$ and a framing c of X with the property $u^i(x,t) = \begin{cases} (v^i(x), 0), \ (x,t) \in N \times [0,\epsilon) \\ (w^i(x), 0), \ (x,t) \in N \times (1-\epsilon, 1] \end{cases}$.

Proposition 5 Framed cobordant is an equivalence relation, which we will denote by $\sim_{fc}$.

As hinted at before, consider the specific case of a map $f : M \to S^n$ and let $y \in S^n$ be a regular value. A given f induces a framing on $f^{-1}(y)$ as follows: choose a positively oriented basis $b = (v^1, \ldots, v^n)$ for $T_y S^n$; recall from a previous section that $df_x : T_x M \to T_y S^n$ has kernel $T_x(f^{-1}(y))$, hence maps $(T_x(f^{-1}(y)))^{\perp} \cong T_y S^n$; given this is a linear isomorphism, $\exists$ a unique basis element $w^i(x) \in (T_x(f^{-1}(y)))^{\perp}$ corresponding to v^i (i.e. $w^i(x) \mapsto v^i$ under f). Notationally, denote the pullback framing on $f^{-1}(y)$ by $m = f^{-1}(b)$.

Definition 18 The framed (sub)manifold $(f^{-1}(y), m)$ is the Pontryagin manifold corresponding to f.

Note that the Pontryagin manifold is only unique up to fixed choice of y and basis b of $T_y S^n$. However, the following theorems give more insight toward their classification.

Theorem 9 If y' is a regular value of f distinct from y and b' is a positively oriented basis of $T_y S^n$ distinct from b, then $(f^{-1}(y), f^* b) \sim_{fc} (f^{-1}(y'), f^* b')$.

Theorem 10 Two smooth maps $f, g : M \to S^n$ are smoothly homotopic $\iff$ their associated Pontryagin manifolds are framed cobordant.

Theorem 11 Any compact, framed submanifold (N, m) of codimension n in M is the Pontryagin manifold for some $f : M \to S^n$.

.

Lemma 7 If b, b' are two positively oriented bases at y of T_yS^n, then $(f^{-1}(y), f^*b) \sim_{fc} (f^{-1}(y), f^*b')$. (This proves half of Theorem A.)

Proof. We may consider $b, b' \in GL_n^+(\mathbb{R})$. We know that $GL_n^+(\mathbb{R})$ is a path-connected space, so we can construct a smooth path between b and b'. □

Lemma 8 If y is a regular value of f and z is a point "close" to y, then $f^{-1}(y) \sim_{fc} f^{-1}(z)$.

Proof. Let $C = \{\text{critical points of } f\}$; the set $f(C)$ is compact. Hence there is some $\epsilon > 0$ such that $B_d(y;\epsilon)$ contains only regular values. Given $z \in B_d(y;\epsilon)$, choose a an isotopy of rotations $r_t : S^n \to S^n$ such that $r_1(y) = z$ and (1) $r_t = Id$ for $t \in [0, \epsilon)$, (2) $r_t = r_1$ for $t \in (1-\epsilon, 1]$, and (3) $r_t^{-1}(z)$ lies on the great circle from y to z (thus is a regular value).

Now, define a homotopy $F(x,t) = f_t f(x)$. Then for each t, z is a regular value of $r_t \circ f$, hence of F. Thus $F^{-1}(z)$ is a framed manifold and serves as a framed cobordism between $f^{-1}(z)$ and $(r_1 \circ f)^{-1}(z) = f^{-1}r_1^{-1}(z) = f^{-1}(y)$, which was to be shown. □

Lemma 9 If f is smoothly homotopic to g and a common regular value of these maps, then $f^{-1}(y) \sim_{fc} g^{-1}(y)$.

Proof. Because f is homotopic to g, there is a homotopy $F : M \times [0,1] \to S^n$ between them with $F(x,t) = f(x)$ for $t \in [0,\epsilon)$, $= g(x)$ for $t \in (1-\epsilon, 1]$. Choose a regular value of F which is sufficiently close to y such that the result of the previous Lemma applies and $f^{-1}(y) \sim_{fc} f^{-1}(z)$ and $g^{-1}(y) \sim_{fc} g^{-1}(z)$ (permissible by Sard). Then $F^{-1}(z)$ is a framed manifold which serves as a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$, and in turn $f^{-1}(y) \sim_{fc} g^{-1}(y)$ as we wished to prove. □

Now, having taken these small developmental steps, we can make the leap and prove Theorem 9.

Proof. (9) Let $y, z \in S^n$ be regular values of $f : M \to S^n$. We can choose rotations $r_t : S^n \to S^n$ with the property that $r_0(t) = Id_{S^n}$, and $r_1(y) = z$. This provides a homotopy between f and $r_1 \circ f$, hence Lemma 10 indicates that $(r_1 \circ f)^{-1}(z) = f^{-1}(y) \sim_{fc} f^{-1}(z)$. □

Focusing more narrowly, let M be a manifold of dimension m which is connected and oriented. If N is a framed submanifold of codimension m, then N is necessarily a finite set of points with a specified 'basis' at each. Let $sign(x) = \pm 1$ depending on if the basis at a given point gives the right or wrong orientation. But recalling the definition of the Brouwer degree, we know $\sum sign(x) = \sum sign(df_x) = deg(f)$. Moreover, it is possible to convince oneself of the fact that the framed cobordism class of N is completely determined by the sum $\sum sign(x)$.

Theorem 12 (Hopf) If M is a manifold which is connected, oriented, and boundaryless, $f, g : M \to S^m$ are homotopic $\iff$ $deg(f) = deg(g)$. (Analogous statement holds for nonorientable manifolds with deg replaced by deg_2.)

6 Lefschetz Fixed Point Theorem

When we first defined the integer degree of a map $f : M \to N$, we held the following assumptions: M, N both oriented; $\dim M = \dim N$; M compact; N connected. Now we will outline a slightly more general framework to allow for broader purposes for the moment: let M be a compact, boundaryless manifold, let N also be boundaryless and have $N' \subset N$ be a closed, boundaryless submanifold such that $\dim M + \dim N' = \dim N$. Then provided $f : M \to N$ is transversal to N', dimensional considerations along with the facts that N' is closed and M is compact imply $|f^{-1}(N')| < \infty$, and each $x \in f^{-1}(Z)$ is endowed with the preimage orientation. In this setup, we refer to the sum of these orientation numbers (all ± 1 as is standard by now for 0-manifolds) as the intersection number, $I(f, N')$. Notice if we re-impose our earlier restrictions and consider an arbitrary map $f : M \to N$ where N is connected and has the same dimension as M, $I(f, \{y\}) =: deg(f)$

is the indersection number of f with any point y; however, as before, we know this is independent of y.

We may now work with instances where M may be considered a submanifold of N, i.e. where $f = i$ is the inclusion of M into N. Here, once again assuming N' is a submanifold of N with appropriate properties, we define $I(M, N') := I(i, N')$. If M is transversal to N', $I(M, N')$ is determined by counting the points in $M \cap N'$, assigning a $+1$ to those points where the orientation of M 'plus' the orientation of N' gives the orientation of N, and a -1 is assigned otherwise. Note here that the order in which we consider the orientations of M and N' *clearly* matters!

To further generalize, we would like to somewhat remove N' from N and then show $I(M, N')$ is independent of perturbations in the second slot (as we have seen for the first).

Proposition 6 $I(f, g) = (-1)^{\dim M \cdot \dim N'} I(g, f)$.

Proof. Recall that we have the direct sum orientation resulting from $T_yN = df_xT_xM \oplus dg_zT_zN'$, but we could just as easily express an orientation resulting instead from $T_yN = dg_zT_zN' \oplus df_xT_xM$. To rearrange one such orientation to give the other requires that we make $\dim M \cdot \dim N'$ transpositions of basis elements. Recalling our assignment of -1 at a point when the direct sum orientation did not add up to the orientation of N, each such transposition yields a factor of -1. Thus $I(f, g) = (-1)^{\dim M \cdot \dim N'} I(g, f)$. □

Corollary 12.1 When $M, N' \subset N$ are compact submanifolds, $I(M, N') = (-1)^{(\dim M \cdot \dim N')} I(N', M)$.

The last formula leads to an interesting result. Consider the case where M is a submanifold of N and $\dim M = \frac{1}{2} \dim N$ such that $I(M, M)$ is a defined quantity. If $\dim M$ is odd, then of course $I(M, M) = (-1)^{\dim M \cdot \dim M} I(M, M) = -I(M, M) \implies I(M, M) = 0$. In turn, $I_2(M, M) = I(M, M) \mod 2 = 0$ too. Mod 2 intersection theory makes no reference to orientability, so it is well-defined regardless of the orientability of the ambient manifold N. Therefore if $I_2(M, M) \neq 0$ or $I(M, M) \neq 0$, N must be nonorientable! Example: the Möbius strip as N and the central circle as M.

Suppose N is compact and oriented; then the <u>Euler characteristic $\chi(N)$</u> is defined as $\chi(N) = I(\Delta, \Delta)$ where Δ is the diagonal in $N \times N$.

Proposition 7 If M is an odd dimensional, compact, oriented manifold, $\implies \chi(M) = 0$.

Proof. From the previous result and definition, $\chi(M) = I(\Delta, \Delta)$ where Δ is the diagonal in $M \times M$. We know that $\Delta \approx^D M$, so in particular $\dim \Delta = \dim M$ is odd. Therefore $I(\Delta, \Delta) = (-1)^{\dim \Delta \cdot \dim \Delta} I(\Delta, \Delta) = -I(\Delta, \Delta) \implies I(\Delta, \Delta) = 0 \implies \chi(M) = 0$. Specifically, we can then state that $\chi(S^{2k+1}) = \chi(T^{2m+1}) = 0$, $k, m \in \mathbb{Z}^+$. □

A very natural idea (as Brouwer would agree) is to count the number of fixed points of a smooth map $f : X \to X$ where X is a compact, oriented manifold. This can be restated equivalently in two relevant and/or illuminative ways: How many solutions are there to the equation $f(x) = x$? Or, phrased slightly differently, how can we compute $|graph(f) \cap \Delta|$? Note immediately that $graph(f)$ and Δ are submanifolds of $X \times X$ satisfying the appropriate hypotheses of intersection theory so we may begin to answer these questions.

Definition 19 Let $f : X \to X$ be a smooth map from a compact, oriented manifold X to itself. Then we define the <u>global Lefschetz number of f</u> to be $I(\Delta, graph(f)) = L(f)$.

Theorem 13 (Smooth Lefschetz Fixed-Point) Let $f : X \to X$ be smooth, and suppose X is a compact, orientable manifold. If $L(f) \neq 0$, $\implies f$ has a fixed point.

Proof. Suppose f has no fixed points; then from the generic description in the preceding paragraphs we know $graph(f) \cap \Delta = \emptyset \implies graph(f) \pitchfork \Delta$ vacuously. In turn, $L(f) = I(\Delta, graph(f)) = 0$. This proves ($f$ has no fixed points) $\implies$ ($L(f) = 0$) which is logically equivalent to the statement of the theorem. □

Proposition 8 $L(f)$ is a homotopy invariant.

This has some relatively simple intuition behind it: $graph(Id_X) = \Delta$, hence $L(Id_X) = I(\Delta, graph(Id_X)) = I(\Delta, \Delta) =: \chi(X)$, and the Euler characteristic is perhaps the most famous of all topological invariants. To approach the full-fledged proposition given above, we first state

Proposition 9 If f is homotopic to Id_X, then $L(f) = \chi(X) = L(Id_X)$. [Specifically, if there is a map $f : X \to X$, $f \sim Id_X$ and $L(f) = 0 \implies \chi(X) = 0$.]

Because the theory we have developed works best for transversal submanifolds, we first look at maps $f : X \to X$ satisfying $graph(f) \pitchfork \Delta$, i.e. *Lefschetz maps.* Hearkening back to the theorem stating any map is homotopic to one which is transversal, it is not hard to convince oneself that a similar result holds in this context.

Proposition 10 Every map $f : X \to X$ is homotopic to a Lefschetz map.

Proof. As the definition of a Lefschetz map hinges upon transversality, well-known facts about transversality give this result. □

We may approach the notion of a Lefschetz map from a slightly different angle. Take x to be a fixed point of f, i.e. $(x, f(x)) = (x, x) \in graph(f) \cap \Delta$. The definition of transversality is phrased in this context as $graph(df_x) + \Delta_x = T_xX \times T_xX$. Because the quantities being summed on the LHS are vector subspaces of the RHS and have complementary dimension, they span the RHS exactly when their intersection is zero. But (as seen in homework for 440) $graph(df_x) \cap \Delta_x = 0$ indicates df_x does not have 1 as an eigenvalue. Such a point x is known as a Lefschetz fixed point, and hence f is a Lefschetz map $\iff$ every fixed point of f is Lefschetz.

For x a Lefschetz fixed point, the orientation ± 1 of (x, x) is referred to as the local Lefschetz number of f at x. This leads to the equality $L(f) = \sum_{(x,x) \in graph(f) \cap \Delta} L_x(f)$. Perhaps an even more computationally useful result is found when we note that if $I : T_xX \to T_xX$ is the identity map, then the condition of x being a Lefschetz point is the same as $df_x - I$ being a linear isomorphism of T_xX. That is, the kernel of $df_x - I$ must be trivial because $Ker(df_x - I) = \{v \in T_xX \,|\, df_x(v) - v = 0\} = \{v \in T_xX \,|\, df_x(v) = v\}$ =the fixed point set of df_x.

Proposition 11 $L_x(f)$ at a Lefschetz fixed point x is ± 1 according to whether $df_x - I$ preserves or reverses the orientation on T_xX ($\iff \det(df_x - I) > 0$ or < 0).

Proof. Let $\beta = \{v_1, \ldots, v_n\}$ be a positively oriented basis for T_xX. Then we obtain positively ordered bases for $T_{(x,x)}\Delta, T_{(x,x)}(graph(f))$, namely $\{(v_1, v_1), \ldots, (v_n, v_n)\}$ and $\{(v_1, df_x(v_1)), \ldots, (v_n, df_x(v_n))\}$. Therefore, $sign(L_x(f)) = sign\{(v_1, v_1), \ldots, (v_n, v_n), (v_1, df_x(v_1)), \ldots, (v_n, df_x(v_n))\}$. Altering by linear combination of basis vectors doesn't change orientation, so the previous sign is equal to the sign of $\{(v_1, v_1), \ldots, (v_n, v_n), (0, (df_x - I)v_1), \ldots, (0, (df_x - I)v_n)\}$. Moreover, given that $df_x - I$ is a linear isomorphism we can repeat in a similar fashion and note the orientation of $\{(v_1, 0), \ldots, (v_n, 0), (0, (df_x - I)v_1), \ldots, (0, (df_x - I)v_n)\} = \{\beta \times 0, 0 \times (df_x - I)\beta\}$ is the same as the previous two. From the product orientation, the sign of this is simply $sign(\beta)sign((df_x - I)\beta)$, which shows that $sign(df_x - I)$ determines the sign of the whole orientation. □

Example: Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ to be a map which fixes $\vec{0}$, and let $A = df_0$. Then we may express $f(x) = Ax + \epsilon(x)$. Assuming A has two independent (real) eigenvectors, it may be written $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. Then, based on the previous proposition, $L_0(f) = sign(\det(A - I)) = sign[(a_1 - 1)(a_2 - 1)]$. Now we may break into cases, assuming $a_i > 0$.
Case 1: $a_i > 1$ for all i. Then $L_0(f) = +1$, and since both eigenvalues of A are > 1 we know all vectors about the origin are being stretched. Hence we have a source.
Case 2: $a_i < 1$ for all i. Then as before, $L_0(f) > 0$, but now $0 < a_i < 1$ for $i = 1, 2$, i.e. the eigenvalues of vectors about the origin serve to shrink or contract. Hence we have a sink.
Case 3: WLOG $0 < a_1 < 1 < a_2$. Now we finally see $L_0(f) = -1$, and here the eigenvalues stretch vectors in the direction of one axis and contract along the second axis. Hence the origin becomes a saddle point.

In this fashion, $L_x(f)$ gives a qualitative description of the topological function behavior near the fixed point in question. As a more interesting particular example, let $f : S^2 \to S^2$ be a map which 'slides' all points except $N = (0,0,1)$ and $S = (0,0,-1)$ toward S. Then from a simple sketch we can see $L_N(f) = L_S(f) = +1$ (the fixed point at the north pole being a source, the south pole being a sink by construction). Hence $L(f) = \sum_{f(x)=x} L_x(f) = 2$. Clearly though a homotopy could be constructed between f and the identity, so from a previous theorem $L(f) = L(Id_{S^2}) = \chi(S^2) = 2$.

Corollary 13.1 The Euler characteristic of the 2-sphere is 2.

Corollary 13.2 Every map $f : S^2 \to S^2$ such that $f \sim Id_{S^2}$ must possess a fixed point. In particular then the map $a(x) = -x$ cannot be homotopic to the identity.

Proof. Suppose such a map f did not possess a fixed point; then by the first theorem in this section, $L(f) = 0$. However, since we specified $f \sim Id_{S^2}$ we must also have $L(f) = L(Id_{S^2}) = \chi(S^2) = 2$, i.e. $0 = 2$, a contradiction. □

6.1 Gauss-Bonnet Theorem

As another striking example of the power of degree and intersection theory, interesting results involving integration of differential forms arise. Throughout this section the definitions which give structure to the exterior algebra of forms are assumed without statement.

Degree Formula: Let $f : X \to Y$ be a smooth map between compact, oriented manifolds, each of dimension k. Let ω be a top dimensional form on Y. Then $\boxed{\int_X f^*\omega = deg(f) \int_Y \omega}$. Thus pulling back a form alters its integral by an integer multiple which is determined entirely by the topology of X and Y.

Theorem 14 If $X = \partial W$, $f : X \to Y$ extends to $F : W \to Y$, then $\int_X f^*\omega = 0 \; \forall \omega \in \Lambda^k(Y)$. Crucially here all manifolds are compact and $\dim X = \dim Y = k$.

Proof. The steps are almost computational, invoking hypotheses as we go:

$$\int_X f^*\omega \underbrace{=}_{X=\partial W} \int_{\partial W} f^*\omega \underbrace{=}_{F\big|_X=f} \int_{\partial W} F^*\omega \underbrace{=}_{Stoke's} \int_W F^*d\omega \underbrace{=}_{d\omega=0} 0.$$

□

Corollary 14.1 If $f_0, f_1 : X \to Y$ are homotopic maps, then for any $\omega \in \Lambda^k(Y)$, $\int_X f_0^*\omega = \int_X f_1^*\omega$.

Proof. Given the f_i's are homotopic, we have a homotopy $F : X \times I \to Y$; note $\partial(X \times I) = (X \times \{0\}) - (X \times \{1\}) := X_0 - X_1$. Invoking the conclusion of the previous theorem, $\int_{\partial(X\times I)} (\partial F)^*\omega = 0$, but we also know $\int_{\partial(X\times I)} (\partial F)^*\omega = \int_{X_0} f_0^*\omega - \int_{X_1} f_1^*\omega \implies \int_{X_0} f_0^*\omega = \int_{X_1} f_1^*\omega$ (since $X_i = X$, we are done). □

The following Lemma proves the theorem locally; extending to the full manifold amounts to utilizing an isotopy h to cover Y, so we will settle for proving the Lemma (for now).

Lemma 10 Let $y \in Y$ be a regular value of $f : X \to Y$. Then there is a neighborhood V of $y$such that the Degree Formula holds for every k-form ω supported in U.

Proof. f is a diffeomorphism about each point in $f^{-1}(y)$, so y has a neighborhood V such that $f^{-1}(V) = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_l$ and $U_i \approx^D V$ for all i. From what we know about pullbacks, (ω has support in $V \implies f^*\omega$ has support in $f^{-1}(V)$. Therefore $\int_{f^{-1}(V)} f^*\omega = \sum_{i=1}^{l} \int_{U_i} f^*\omega$. Given f is a diffeomorphism between each of the U_i and V, $\int_{V_i} f^*\omega = sign_i(f) \int_U \omega$ where the sign of f is determined based on whether it is an orientation preserving or reversing diffeomorphism. Thus

$$\int_{f^{-1}(V)} f^*\omega = \sum_{i=1}^{l} sign_i(f) \int_U \omega =: deg(f) \int_U \omega.$$

□

Recall the definition of the Gauss map, $g : X \to S^k$, $x \mapsto \hat{n}_x$ (changing to G& P notation here for unit normal vector at x). The Jacobian of this map is the <u>curvature</u> of X at x, $J(g) = \kappa(x)$. This data associated with X is strictly geometric (unlike the many topological structures we have seen), so it is not preserves by typical topological transformations. We can however say something beautiful about the total curvature, i.e. $\int_X \kappa$.

Gauss-Bonnet Theorem: If X is a compact, even-dimensional manifold embedded in $\mathbb{R}^{k+1}$ (a hypersurface), then

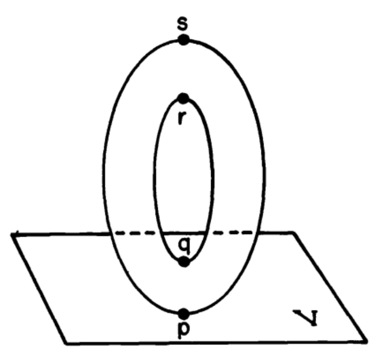
$$\int_X \kappa = \frac{1}{2} Vol(S^k)\chi(X).$$

It is important to make note of the fact that the 'even-dimensional' hypothesis is of utmost importance; we saw in the previous section that if X were odd dimensional, it is automatic that $\chi(X) = 0$, but the integral of the curvature may not be.

7 Morse Theory

The previous sections give us a sense that topology may be approached through a variety of methods, a rather pleasant surprise. If possible we would like to know to what extend the topology of a manifold can be described by studying maps $f : M \to \mathbb{R}$. The amount of information which this theory, Morse theory, has to offer is tremendous. We introduce some of the main concepts and goals of through the following motivating example.

Example: Let $M = T^2$ as depicted below, tangent to the plane V.



Let $f : M \to \mathbb{R}$ be the height above V, and denote by $M^a = \{x \in M \mid f(x) \leq a\}$. What does M^a look like for varying a?

(1) $a < f(p) := 0$. Then $M^a = \emptyset$.
(2) $f(p) < a < f(q)$. Then $M^a \sim e^2$ (e^k is the standard k-cell).
(3) $f(q) < a < f(r)$. Then $M^a \sim S^1 \times (0, 1)$ (an open cylinder).
(4) $f(r) < a < f(s)$. Then M^a is $T^2 - e^2$, i.e. a torus with a disk removed. This is a compact 2-manifold with boundary S^1.
(5) $a > f(s)$. Then (quite clearly) M^a is all of M.

Note that already *homotopy type* takes precedence here over *homeomorphism type*. In that vein, the progression from (1)-(5) above can be viewed as

$$(1) \underbrace{\to}_{\text{attach a 0-cell}} (2) \underbrace{\to}_{\text{attach a 1-cell}} (3) \underbrace{\to}_{\text{attach a 1-cell}} (4) \underbrace{\to}_{\text{attach a 2-cell}} (5)$$

This process of 'attaching' cells is terribly vague; let's make it more precise.
Let X be a topological space and let $e^k = \{x \in \mathbb{R}^k \,|\, |x| \leq 1\}$, $\partial e^k = \{x \in \mathbb{R}^k \,|\, |x| = 1\} =: S^{k-1}$. If $f : S^{k-1} \to X$ is continuous, then X with a k-cell attached via f, indicated by $X \cup_f e^k$, is $X \sqcup e^k / x \sim g(x)$ where the equivalence relation is defined on $x \in S^{k-1}$. Note: convention dictates that $e^0 = \{*\}$, $\partial e^0 = \emptyset$.

The points p, q, r, and s at which the classification of M^a up to homotopy changes are the critical points of the function f. As the base case, consider a neighborhood about the point p; intuitively, we may choose a coordinate system (x, y) about p and contained in this neighborhood such that $f(x, y) = x^2 + y^2$ (the bottom of the torus looks locally like the bottom of a paraboloid). In turn, assuming we have centered p at the origin of our coordinate system which we may do via a diffeomorphism, $\nabla f_p = (2x, 2y)_p = (0, 0)$. In a similar fashion at s our local coordinates are such that $f(x, y) = -x^2 - y^2$, while around q and r $f(x, y) = C + x^2 - y^2$. Something worth paying attention to here is that *the number of minus signs in the local expression of f about each critical point is precisely the dimension of the cell being attached.* These notions will be formalized forthwith, first, some familiar definitions.

Definition 20 Let $f : M \to \mathbb{R}$ be smooth. A point $p \in M$ is a critical point of f if the linear map $df_p : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is 0. Locally, this is expressed by the condition $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i = 1, \ldots, n$, i.e. $\nabla f_p = \vec{0}$. $f(p)$ is a critical value.

Definition 21 Let p be a critical point of f. The point p is non-degenerate when the Hessian $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]$ is nonsingular, or has nonzero determinant.

Definition 22 The index of the Hessian H, a symmetric bilinear functional in a certain light, is the maximum dimension of a subspace $V \subseteq T_pM$ on which H is negative definite. The nullity is the dimension of the null space of H. In particular then, a critical point p is nondegenerate $\iff$ the nullity of H_p is 0.

With this basic groundwork in place we can work toward stating and proving the Lemma of Morse.

Lemma 11 Suppose f is a smooth function in a convex neighborhood V of the origin in $\mathbb{R}^n$ such that $f(0) = 0$. Then $f(x_1, \ldots, x_n) = \sum x_i g_i(x_1, \ldots, x_n)$, where the functions g_i are smooth, defined in V, and satisfy $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Proof.

$$f(x_1, \ldots, x_n) \underbrace{=}_{\text{FTC}} \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n) dt \underbrace{=}_{\text{chain rule}} \int_0^1 \sum \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) \cdot x_i dt.$$

From this simple manipulation, we see to let $g_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) dt$. □

Lemma of Morse: If p is a nondegenerate critical point of f, then there is a local coordinate system $(x_1, \ldots, x_n)$ inside a neighborhood U of p such that
(i) $x_i(p) = 0$ for all i, and
(ii) the identity $f(x_1, \ldots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ for all $x \in U$, where λ is the index of the critical point p.

Proof. We only prove that λ is the index of p. So, suppose such a coordinate expression exists for the function f, $(x_1, \ldots, x_n)$. We want to show that the index of f at the critical point p must then be λ. We have at any point $q \in U$ that $f(q) = f(p) - (x_1(q))^2 - \cdots - (x_\lambda(q))^2 + (x_{\lambda+1}(q))^2 + \cdots + (x_n(q))^2$. Then, computing an arbitrary second partial we know

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(p) = \begin{cases} 2, \ i = j \geq \lambda \\ -2, \ i = j \leq \lambda \\ 0, \ \text{otherwise} \end{cases} \implies H_p(f) = \begin{bmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{bmatrix}.$$

By inspection of the Hessian, we see there is a subspace $V \subset T_pM$, $\dim V = \lambda$, on which $H_p(f)$ is negative definite (span of first λ columns of H). Similarly, there is a subspace W, $\dim W = n - \lambda$, on which $H_p(f)$ is positive definite (span of last $n - \lambda$ columns of H). From dimensionality, if there were a subspace V' with $\dim V' > \dim V$ on which H was negative definite, $V' \cap W \neq \emptyset$, which is clearly a contradiction. Thus λ is indeed the index of H.

□

Corollary 14.2 Non-degenerate critical points are isolated.

Here it is necessary to introduce groups of diffeomorphisms of a manifold, as they will be used in a subsequent proof.

Definition 23 A <u>1-parameter group of diffeomorphisms</u> of M is a family of maps $\varphi_t : \mathbb{R} \times M \to M$ satisfying
(i) for each fixed value of $t \in \mathbb{R}$, $\varphi_t : M \to M$, $\varphi_t(p) = \varphi(t, p)$ is a diffeomorphism from M to itself.
(ii) $\forall t, s \in \mathbb{R}$, $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

If $\{\phi_t\}$ is a 1-parameter group of diffeomorphisms on a manifold M, we can define a corresponding vector field X. For every smooth $f : M \to \mathbb{R}$, $X_p(f) := \lim_{h \to 0} \frac{f(\varphi_h(p)) - f(p)}{h}$. Then, X 'generates' $\{\varphi_t\}$.

Lemma 12 A smooth vector field $X \in \mathfrak{X}(M)$ which vanishes outside a compact set $C \subseteq M$ generates a unique 1-parameter diffeomorphism group.

We can now turn to the more prominent results of Morse theory, once we recall the following definitions from basic topology. From here forward, unless otherwise specified, let $\boxed{M^a = f^{-1}((-\infty, a])} = \{p \in M \mid f(p) \leq a\}$.

Definition 24 Let $A \subseteq X$ be a subspace of a topological space. A <u>deformation retract</u> of X onto A is a collection of maps $f_t : X \to X$ with the property that $f_0 = Id_X$, $f_1(X) = A$, and $f_t\big|_A = Id_A$ for all values of t. The map $f_t : X \times I \to X$, $(x, t) \mapsto f_t(x)$ must be continuous.

Definition 25 Let X, Y be topological spaces. We say X is <u>homotopy equivalent</u> to Y if there is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that the compositions $g \circ f \sim Id_X$, and $f \circ g \sim Id_Y$. We denote this (equivalence) relation by $X \sim Y$ where it is clear from context when $\sim$ indicates a homotopy equivalence of spaces versus homotopy equivalence of maps.

Theorem 15 Let $f : M \to \mathbb{R}$ be smooth, $a < b \in \mathbb{R}$, and suppose $f^{-1}([a, b])$ is compact, containing only regular points of f. Then $M^a \approx^D M^b$. In particular, M^a is a deformation retract of M^b and hence the two are homotopy equivalent.

Proof. Moderately lengthy proof; the idea is to 'push' M^b to M^a through the level sets $f^{-1}(\{c\})$, c varying from b to a. To do this, first let $g : TM \times TM \to \mathbb{R}$ be a Riemannian metric on M, where by $\langle X, Y \rangle_p =: g_p(X, Y)$ is the inner product of $X, Y \in T_pM$ as determined by g. The *gradient* of f, $grad\, f$, is the vector field which satisfies $\langle X, grad\, f \rangle = X(f)$ for any X. (Note: in the case $M = \mathbb{R}^n$, we have the familiar $grad\, f = \nabla f = (\partial_1 f, \ldots, \partial_n f)$. We know in this case $grad\, f(p) = \vec{0} \iff p$ is a critical point of f. The same holds true in the case of general M, where in local coordinates $(x_1, \ldots, x_n)$ the components of $grad\, f$ look like $\sum g^{ij} \frac{\partial f}{\partial x_j}$, hence overall $grad\, f = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$.) For $c : \mathbb{R} \to M$ a curve, note we have the identity

$$\left\langle \frac{dc}{dt}, grad\, f \right\rangle = \frac{d(f \circ c)}{dt}.$$

In the case of $M = \mathbb{R}^n$, this is a simple application of the multivariable chain rule as $\frac{d(f \circ c)}{dt} = \sum \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} = \left\langle \frac{dc}{dt}, \nabla f \right\rangle$. Keep this **Fact** in mind. Now, define $\rho : M \to \mathbb{R}$ be smooth and

$$\rho(p) = \begin{cases} \frac{1}{\langle grad\, f, grad\, f \rangle}, & p \in f^{-1}([a, b]) \\ 0, & \text{"else"} \end{cases}.$$

Then we can define $X_q = \rho(q)(grad\, f)$ which is a vector field satisfying the conditions of the previous Lemma. As a consequence, we obtain a 1-parameter group of transformations $\{\varphi_t\}$, $\varphi_t : M \to M$.

For a fixed value $q \in M$, consider the map given by $t \mapsto f(\varphi_t(q))$. Provided $\varphi_t(q) \in f^{-1}([a,b])$, the **Fact** from before says $\frac{d(f\circ\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, grad\, f \right\rangle = \langle X, grad\, f \rangle = 1$. Thus we see this map is linear, with derivative 1 for $f(\varphi_t(q))$ between a and b.

Next, from this family $\{\varphi_t\}$, consider the diffeomorphism φ_{a-b}; Clearly this carries M^a diffeomorphically to M^b.

Lastly, define a 1-parameter family $r_t : M^b \to M^a$,

$$r_t(q) = \begin{cases} q, \ f(q) \leq a \\ \varphi_{t(a-f(q))}(q), \ \ a \leq f(q) \leq b \end{cases} .$$

We see $r_0 = Id_{M^b}$, $r_1(M^b) = M^a$, $r_t\big|_{M^a} = Id_{M^a}$, and hence is a deformation retract $M^b \to M^a$. This concludes the proof. □

Theorem 16 Let $f : M \to \mathbb{R}$ be smooth with $p \in M$ a nondegenerate critical point of f with index λ. If $f(p) := c$, suppose $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains only regular points besides p. Then for sufficiently small $\epsilon > 0$, $M^{c+\epsilon}$ hs the homotopy type of $M^{c-\epsilon}$ with a λ-cell (e^λ) attached.

Corollary 16.1 Suppose we have k nondegenerate critical points p_i, $i = 1, \ldots, k$of indices $\lambda_1, \ldots, \lambda_k$. Then a similar proof to the theorem (carefully iterated) shows $M^{c+\epsilon} \sim M^{c-\epsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$.

Theorem 17 Let $f : M \to \mathbb{R}$ be smooth and devoid of degenerate critical points, and having each M^a compact, then M has the homotopy type of a CW-complex with a λ-cell for each critical point of index λ.

To prove this theorem, we prove two lemmas.

Lemma 13 Let $\varphi_0 \sim \varphi_1$ be homotopic maps $\partial e^\lambda \to X$. Then Id_X extends to a homotopy equivalence

$$\Phi : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda.$$

Proof. We need to construct Φ as an extension of Id_X. Let

$$\Phi = \begin{cases} \Phi(x) = x, \quad x \in X \\ \Phi(tu) = 2tu, \quad t \in [0, 1/2], \ u \in \partial e^\lambda \\ \Phi(tu) = \varphi_{2-2t}(u), \quad t \in [1/2, 1], \ u \in \partial e^\lambda \end{cases} .$$

We can construct the 'reverse' map $\Phi_1 : X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_0} e^\lambda$ analogously. □

Lemma 14 Suppose $\varphi : \partial e^\lambda \to X$ is an attaching map; then any homotopy equivalence $f : X \to Y$ may be extended to a homotopy equivalence $F : X \cup_\varphi e^\lambda \to Y \cup_\varphi e^\lambda$.

Proof. (Theorem) Let $\{c_i\}_{i \in I \subseteq \mathbb{Z}_+}$ be the critical values of $f : M \to \mathbb{R}$, ordered by $c_i < c_{i+1}$ for all i. Recall that we are requiring each M^a to be compact, so the sequence of critical values cannot have an accumulation point. Let $a \in \mathbb{R}$, $a \neq c_i$ for any i, and suppose M^a has the homotopy type of a CW-complex. Let $c = c_j$ for some j be the smallest critical value greater than a. By previous results we then know for sufficiently small epsilon $M^{c+\epsilon}$ has the homotopy type of

$$M^{c-\epsilon} \cup_{\phi_1} e^{\lambda_1} \cup_{\phi_2} \cdots \cup_{\phi_{r(c)}} e^{\lambda_{r(c)}}$$

where here we go so far as to specify the attaching maps of each λ-cell. We also have seen that there is a homotopy equivalence $h : M^{c-\epsilon} \to M^a$. We also assumed a homotopy equivalence existed $h' : M^a \to C$ for some CW-complex C.

Then, the composition $\partial e^{\lambda_i} \xrightarrow{\phi_i} M^{c-\epsilon} \xrightarrow{h} M^a \xrightarrow{h'} C$, i.e. $h' \circ h \circ \phi_i$, is homotopic to $\beta_i : \partial e^\lambda \to X^{\lambda_i - 1}$ (where X^{λ_i-1} is the $(\lambda_i - 1)$-skeleton of C). In turn,

$$C \cup_{\beta_1} e^{\lambda_1} \cup_{\beta_2} \cdots \cup_{\beta_{r(c)}} e^{\lambda_{r(c)}} \sim M^{c+\epsilon}.$$

Inductively we have thus shown M^a for any a as specified above has the homotopy type of a *CW*-complex. If M is compact (as it is in many nice examples) we are done. If M is not compact but all of the $f^{-1}(c_i)$ lie in a single compact M^a then we know M^a is a retract of M, i.e. we are done again. This leaves only the case where we have infinitely many c_i, not all lying in a single M^a. In this instance the inductive process outlined above results in an infinite sequence of CW-complexes $C_1 \subset C_2 \subset \cdots$ corresponding to an infinite sequence $M^{a_1} \subset M^{a_2} \subset \cdots$ where we have homotopy equivalence at each stage $M^{a_s} \to C_s$. Taking $C = \cup C_k$ in the direct limit topology and applying a result of Whitehead, the limit map $g : M \to C$ is a homotopy equivalence. □

7.1 Morse Inequalities

The above structure developed concerning the critical points of a real valued function $f : M \to \mathbb{R}$ and how they determine the topology of M was not entirely available to Morse. Instead, this relationship was formulated in terms of certain inequalities which we aim to construct here.

Definition 26 Let f be a function from pairs of spaces to $\mathbb{Z}$. We say f is subadditive if $Z \subset Y \subset X \implies f(X,Z) \leq f(X,Y) + f(Y,Z)$. If equality holds, f is additive.

Example: The Betti numbers in relative homology with coefficients in a field k. Denote the λ^{th} Betti number of the pair (X,Y) by $b_\lambda(X,Y)$ (assuming that rank over F of $H_\lambda(X,Y;F)$ is finite). Suppose $Z \subset Y \subset X$; consider the exact sequence

$$\cdots \to H_\lambda(Y,Z) \to H_\lambda(X,Z) \to H_\lambda(X,Y) \to \cdots .$$

Lemma 15 Let f be subadditive and let $X_0 \subset \cdots \subset X_n$. Then $f(X_n, X_0) \leq \sum f(X_i, X_{i-1})$.

Proof. We proceed by induction on n. The base case $n = 1$ says $f(X_1, X_0) \leq f(X_1, X_0)$, which is obviously satisfied (in fact equality holds). Next, suppose it holds for all $1 \leq n \leq k-1$, and we must show it holds for $n = k$. The inductive hypothesis tells us

$$f(X_{k-1}, X_0) \leq \sum_{i=1}^{k-1} f(X_i, X_{i-1})$$

so considering the sum

$$\sum_{i=1}^{k} f(X_i, X_{i-1}) \geq \sum_{i=1}^{k-1} f(X_i, X_{i-1}) + f(X_k, X_{k-1}) \geq f(X_{k-1}, X_0) + f(X_k, X_{k-1})$$
$$\geq f(X_k, X_0).$$

In the case $X_0 = \emptyset$, we have simply $f(X_n) \leq \sum f(X_i, X_{i-1})$. Equality holds when f is additive. □

Now we get to apply the concept of subadditivity to the situation outlined before. Let M be a compact manifold and let $f : M \to \mathbb{R}$ with isolated and nondegenerate critical points. Choose $a_1 < \cdots < a_k \in \mathbb{R}$ such that M^{a_i} contains i critical points (i.e. $M^{a_0} = \emptyset$, $M^{a_k} = M$). Then notice the following

$$H_*(M^{a_i}, M^{a_{i-1}}) = H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}})$$

where the equality holds because M^{a_i} and $M^{a_{i-1}} \cup e^{\lambda_i}$ are of the same homotopy type. Note then that using the excision theorem

$$H_*(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) = H_*(e^{\lambda_i}, \partial e^{\lambda_i}) = \begin{cases} R, & * = \lambda_i \\ 0, & \text{else} \end{cases}$$

where R is the coefficient ring. From this result, the previous Lemma, and the fact that $\chi(X,Y) = \sum (-1)^\lambda b_\lambda(X,Y)$, we see something very nice. Consider the nesting $\emptyset \subset M^{a_0} \subset \cdots M^{a_k} = M$ and the case $f(\ ,\ ) = b_\lambda(\ ,\ )$; then

$$b_\lambda(M^{a_k}, M^{a_0}) = b_\lambda(M) \leq \sum_{i=1}^{k} b_\lambda(M^{a_i}, M^{a_{i-1}}) = c_\lambda$$

where c_λ is the number of critical points of index λ, since by above we know

$$H_i(M^{a_i}, M^{a_{i-1}}) = \begin{cases} R, & * = \lambda_i \\ 0, & \text{else} \end{cases} \implies b_\lambda(M^{a_i}, M^{a_{i-1}}) = \begin{cases} c_\lambda, & i = \lambda \\ 0, & \text{else} \end{cases}.$$

In the case $f = \chi$, which is additive, we see similarly

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = \sum (-1)^i c_i.$$

We combine the previous results into one concise statement

Theorem 18 (Weak Morse Inequalities) If M is a compact manifold and the number of critical points of index λ on M is c_λ, then (1) $b_\lambda(M) \leq c_\lambda$ and (2) $\chi(M) = \sum(-1)^\lambda b_\lambda(M) = \sum(-1)^\lambda c_\lambda$.

Whenever possible, we want to sharpen inequalities. That is possible here based on the next Lemma.

Lemma 16 Define the function F_λ as

$$F_\lambda(X, Y) = b_\lambda(X, Y) - b_{\lambda-1}(X, Y) \pm \cdots \pm b_0(X, Y).$$

Then F_λ is subadditive.

Proof. Consider the exact sequence of vector spaces given by

$$\to V_0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$$

where we denote each homomorphism mapping $\to V_i$ by h_i. Then exactness and rank-nullity imply that $rank(h_0) + rank(h_1) = rank(V_0) \implies rank(h_0) = rank(V_0) - rank(h_1)$. Applying this again to $h_1, \ldots, h_n$ we obtain

$$rank(h_0) = rank(V_0) - rank(V_1) \pm \cdots \pm rank(V_n) \geq 0.$$

Since this computation was done for general vector spaces and homomorphisms, we can apply the result to the case of

$$\cdots \to H_{\lambda+1}(X, Y) \to H_\lambda(Y, Z) \to H_\lambda(X, Z) \to \cdots$$

where the homomorphisms are boundary operators and the ranks are the Betti numbers. Thus if $\partial : H_{\lambda+1}(X, Y) \to H_\lambda(Y, Z)$,

$$rank(\partial) = b_\lambda(Y, Z) - b_\lambda(X, Z) + b_\lambda(X, Y) - b_{\lambda-1}(Y, Z) \pm \cdots \geq 0.$$

Reorganizing, this is equivalent to

$$F_\lambda(Y, Z) - F_\lambda(X, Z) + F_\lambda(X, Y) \geq 0 \implies F_\lambda(X, Z) \leq F_\lambda(Y, Z) + F_\lambda(X, Y).$$

□

With this function F_λ in hand, we may apply it to the nested collection of spaces $\emptyset \subset M^{a_1} \subset \cdots \subset M^{a_k}$ to obtain the Morse Inequalities:

$$(M_1) \qquad F_\lambda(M) \leq \sum F_\lambda(M^{a_i}, M^{a_{i-1}})$$

$$\iff \quad (M_2) \qquad \boxed{b_\lambda(M) - b_{\lambda-1}(M) \pm \cdots \pm b_0(M) \leq c_\lambda - c_{\lambda-1} \pm \cdots \pm c_0}.$$

8 Morse Homology

We want to bring together all of the constructions that we have developed thus far to connect differential topology, in particular the theory of Morse functions on Riemannian manifolds, to form algebraic topological invariants. It will turn out that a Morse (or Morse-Smale) function $f : M \to \mathbb{R}$ on a finite dimensional, compact, Riemannian manifold (M, g) gives rise to a (Morse-Smale-Witten) chain complex generated by the critical points of f. Then, the fundamental result (Morse Homology Theorem) will prove the Morse-Smale-Witten chain complex is isomorphic to the standard singular homology of M. Assume from here on that every manifold is finite dimensional, smooth, and Riemannian.

8.1 Stable and Unstable Manifolds

If previously not specified clearly enough, we define here the gradient vector field of a function f on a Riemannian manifold (M, g) w.r.t. the metric g.

Definition 27 If $f : M \to \mathbb{R}$ is smooth, then the gradient vector field is the unique smooth vector field ∇f such that for all smooth vector fields $V \in \mathfrak{X}(M)$ we have $g(\nabla f, V) = df(V) = V \cdot f$. Equivalently, $\nabla f = \tilde{g}^{-1}(df)$ where $\tilde{g}$ is the isomorphism induced by g between TM and T^*M.

With a vector field in hand, we know from 8.3 that we may obtain a corresponding 1-parameter family or group of diffeomorphisms. We are especially interested in the family $\{\varphi_t\}$ generated by $-\nabla f$. (For basic intuition, one may think of $-\nabla f$ in the case of the height function on T^2 or S^2, in which case we have a vector field pointing downward along the lines of steepest height descent.) This amounts to saying $\frac{d}{dt}\varphi_t(x) = -\nabla f(\varphi_t(x))$. Denote the integral curve by $\gamma_x(t) \equiv \varphi_t(x)$.

Proposition 12 Every smooth function $f : M \to \mathbb{R}$ decreases along its gradient flow lines.

Proof. Simply compute/show $\frac{d}{dt}(f(\gamma_x(t))) = -\|\nabla f(\gamma_x(t))\|^2 \leq 0$. □

Proposition 13 Let $f : M \to \mathbb{R}$ be a Morse function. Then every gradient flow line begins and ends at a critical point, meaning $\forall x \in M$, $\lim_{t\to\pm\infty} \gamma_x(t)$ both exist and are critical points of f.

Proof. Take $x \in M$ and let $\gamma_x(t)$ be the flow line which passes through it. We know since M is compact that flow lines are defined for all real t and that $f(\gamma_x(\mathbb{R}))$ is a bounded subset of real numbers. Given this boundedness condition, we know by the equality found in the previous proposition that $\lim_{t\to\pm\infty} \frac{d}{dt} f(\gamma_x(t)) = -\lim_{t\to\pm\infty} \|\nabla f(\gamma_x(t))\|^2 = 0$. Take $(t_n)_{n\in\mathbb{Z}^+}$ to be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} t_n = -\infty$. We obtain a corresponding sequence $(\gamma_x(t_n))_{n\in\mathbb{Z}^+} \subseteq M$ which (since compactness implies sequential compactness) must have an accumulation point, p. Because we saw $\|\nabla f(\gamma_x(t_n))\| \to 0$ for increasing n, we find p is a (nondegenerate) critical point, hence there is a closed neighborhood U of p in which p is the only critical point. Suppose though that $\lim_{t\to-\infty} \gamma_x(t) \neq p$; then we can form a new sequence $(\tilde{t}_n)_{n\in\mathbb{Z}^+}$ of real numbers which has an accumulation point such that $\lim_{n\to\infty} \tilde{t}_n = -\infty$ and $\gamma_x(\tilde{t}_n) \in U - V$ for an open neighborhood V of p. But this contradicts our choice of the set U hence $\lim_{t\to-\infty} \gamma_x(t) = p$ as desired. A symmetric argument shows the same to be true for $\lim_{t\to\infty} \gamma_x(t) = q$ for some critical point q. □

As mentioned previously, the gradient determines a 1-parameter group of transformations $\{\varphi_t(x)\}$ on M. From the flow we obtain a new structure.

Definition 28 Let $p \in M$ be a nondegenerate critical point of a smooth function f.
(1) The stable manifold of p is $W^s(p) = \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = p\}$.

(2) The unstable manifold of p is $W^u(p) = \{x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = p\}$.

This leads into the first major theorem of the section.

Theorem 19 Let $f : M \to \mathbb{R}$ be a Morse function and let $\dim M = m < \infty$. If $p \in M$ is a critical point, then there is a splitting of the tangent space into $T_pM = T_p^sM \oplus T_p^uM$ such that the Hessian of f is positive definite on T_p^sM and negative definite on T_p^uM. Additionally, we have smooth embeddings $f^s : T_p^sM \to W^s(p)$ and $f^u : T_p^uM \to W^u(p)$ which implies $\dim W^s(p) = m - \lambda_p$, $\dim W^u(p) = \lambda_p$ where λ_p is the index of p.

Lemma 17 In local coordinates, $\nabla f = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$.

Lemma 18 In local coordinates about a critical point, such that $\left\{ \frac{\partial}{\partial x_i} \right\}$ is an orthonormal basis for the tangent space, the differential of the gradient is the Hessian, i.e. $\frac{\partial}{\partial \vec{x}} \nabla f(p) = H_p(f)$.

Lemma 19 In local coordinates with the same orthonormal basis as before, the matrix for the differential of φ_t is given by

$$\frac{\partial}{\partial \vec{x}} \varphi_t(p) = \exp(-H_p(f)t).$$

Proposition 14 If $f : M \to \mathbb{R}$ is Morse, where M is (among the hypotheses outlined at the beginning of the section) closed, then $M = \sqcup_{p \in Crit(f)} W^u(p)$, and/or similarly $M = \sqcup_{q \in Crit(f)} W^s(q)$.

A nice example: Consider the sphere $S^n \subset \mathbb{R}^{n+1}$ and define a map $f : S^n \to \mathbb{R}$ by $f(x_1, \ldots, x_{n+1}) = x_{n+1}$ (if we assume the $n+1$ coordinate is the "vertical" one through the north and south poles N and S, this is the height function). It is clear f has 2 critical points, N and S, of index n and 0 respectively (since local coords about N would look like $-\sum x_i^2$, while about S they would look like $\sum x_i^2$). In turn, we find

$$W^s(N) = \{N\};\ W^u(N) = S^n - \{S\};\ W^s(S) = S^n - \{N\};\ W^u(S) = \{S\}.$$

8.2 Morse-Smale Functions

Definition 29 A Morse function $f : M \to \mathbb{R}$ satisfies Morse-Smale transversality if $W^u(q) \pitchfork W^s(p)$ for all critical points p, q of f. Such a function is simply called Morse-Smale.

From basic differential topology notions, embedded submanifolds $W(q,p) := W^u(q) \cap W^s(p)$ are obtained, and are of dimension $\lambda_q - \lambda_p$. As a consequence we find a Corollary.

Corollary 19.1 If $f : M \to \mathbb{R}$ is a Morse-Smale function, then the index of the critical points decreases strictly along gradient flow lines. In other words, if p, q are critical points of f with $W(q,p) \neq \emptyset$, then $\lambda_q > \lambda$.

Proof. Given $W(q,p)$ is nonempty, it contains at least one flow line from $q \to p$. The dimension of this flow line must be 1, hence $W(q,p) \geq 1$. □

Example Revisited: We return to the simple example of the height function of the torus positioned vertically on a plane. Immediately we notice a problem: the preceding Corollary does not appear to hold here, as the flow lines between the top of the "hole" and bottom of the "hole" connect critical points which are both of index 1, hence the index of the critical points is not strictly decreasing. This is because the standard height function is not Morse-Smale as the flow lines very clearly do not intersect transversely. We can however *tilt* the torus slightly and in so doing perturb the height function into one that is Morse-Smale. Later we will state a theorem concerning the feasability of deforming a function $f : M \to \mathbb{R}$ into one which satisfies these conditions.

Definition 30 A topological space X is separable if it contains a countable, dense subset. A subset $A \subseteq X$ is residual if it is a countable intersection of open, dense subsets of X. A subset of X is generic if it contains a residual set. X is called Baire if every generic subset is dense.

Theorem 20 If (M, g) is finite dimensional, compact, then the set of Morse-Smale gradient vector fields of class C^r is a generic subset of all gradient vector fields on M of class C^r.

Proposition 15 Let M be smooth and suppose p, q are hyperbolic fixed points of $\varphi \in Diff(M)$. If at some point $W^u(q)$ and $W^s(p)$ intersect transversely, then $\overline{W^u(q)} \supseteq W^u(p)$.

Proposition 16 (Transitivity of Gradient Flows) Let p, q, r be critical points of a Morse-Smale function $f : M \to \mathbb{R}$. If $W(r,q), W(q,p) \neq \emptyset, \implies W(r,p) \neq \emptyset$ and $\overline{W(r,p)} \supseteq W(r,q) \cup W(q,p) \cup \{p,q,r\}$.

From this proposition we see the critical points of a Morse-Smale function can be partially ordered.

Definition 31 For critical points p, q of $f : M \to \mathbb{R}$, we say q is succeeded by p, indicated by $q \succeq p$, provided $W(q,p) \neq \emptyset$, i.e. there is at least one gradient flow line from q to p.

Corollary 20.1 If p, q are critical points with $\lambda_q - \lambda_p = 1$, then $\overline{W(q,p)} = W(q,p) \cup \{p,q\}$ and $W(q,p)$ has finitely many components (number of gradient flows from q to p is $< \infty$).

Proof. $W(q,p) \cup \{p,q\}$ since Corollary 19.1 indicates there are no intermediate critical points between q and p. As a result, $W(q,p) \cup \{p,q\}$ is a closed subset of a compact space, hence compact. We know the gradient flow lines between q and p form and open cover of $W(q,p)$ which can be extended to an open cover of $W(q,p) \cup \{p,q\}$. From the definition of compactness we thus see the number of gradient flow lines is finite. □

Definition 32 For p, q critical points of $f : M \to \mathbb{R}$, p is an immediate successor of q if $q \succeq p$ and $\nexists r \neq p$ such that $q \succeq r$ and $r \succeq p$.

Let $f : M \to \mathbb{R}$ be Morse-Smale and assume p is an immediate successor of q. Let $t \in \mathbb{R}$, be a regular value of f between $f(p)$ and $f(q)$. From the Regular Value Theorem the set $f^{-1}(t)$ for all such t is a submanifold of M, $\dim(f^{-1}(t)) = m - 1$, and $f^{-1}(t)$ is transverse to both $W^u(q)$ and $W^s(p)$. This leads to the following definition of two other submanifolds of M.

Definition 33 The unstable sphere of q is $S^u(q) = W^u(q) \cap f^{-1}(t)$, while the stable sphere of p is $S^s(p) = W^s(p) \cap f^{-1}(t)$, which are embedded submanifolds of dimensions $\lambda_q - 1$ and $m - \lambda_p - 1$.

Immediately following from this definition is the fact that $W^u(q) \pitchfork W^s(q) \implies S^u(q) \pitchfork S^s(p) \implies S^u(q) \cap S^s(p) =: N(q,p)$ is an embedded submanifold of dimension $\lambda_1 - \lambda_p - 1$ inside of $f^{-1}(t)$.

8.3 Associated Chain Complex

In this section we bring all of our buildup to a head by defining a chain complex known as the Morse-Smale or occasionally Morse-Smale-Witten chain complex. As with any chain complex, the boundary operator is of fundamental importance. In this instance the boundary operator will be expressed through the intersection numbers of unstable spheres of critical points of index k, and stable spheres of critical points of index $k - 1$.

Again, take (M, g) to be a finite dimensional, compact, smooth, oriented Riemannian manifold, along with $f : M \to \mathbb{R}$ a Morse-Smale function such that $W^u(q) \pitchfork W^s(p)$ for all critical points p, q of f. Let $Cr_k(f)$ be the set of critical points p of f for which $\lambda_p = k$, and more broadly let $Cr(f)$ be the set of all critical points of the function. For $p \in Cr(f)$, form a basis β^u_p of $T_pW^u(p)$ which in turn gives the space an orientation. Moreover by transversality we know $T_pM = T_pW^s(p) \oplus T_pW^u(p)$, hence to be compatible with the orientation of the full vector space our choice of basis for $T_pW^u(p)$ determines an orientation of $T_pW^s(p)$ (i.e. $W^u(p)$ and $W^s(p)$ have orientations compatible with M at the point p).

Now take $p, q \in Cr(f)$ of index $\lambda_p = k - 1$ and $\lambda_q = k$ respectively, while assuming $q \succeq p$. Letting $\gamma : \mathbb{R} \to M$ be the gradient flow from q to p we recall $\dot{\gamma}(t) = -(\nabla f)(\gamma(t))$, $\lim_{t \to -\infty} \gamma(t) = q$, and $\lim_{t \to \infty} \gamma(t) = p$. For any $x \in \gamma(\mathbb{R})$ inside $W(q,p)$ we may add vectors to $-(\nabla f)(x)$ to complete a positive basis $(-(\nabla f), \hat{\beta}^u_x)$ for $T_xW^u(q)$. Along with a positive orientation β^s_x (which does not need to be completed) we form a basis $(\beta^s_x, \hat{\beta}^u_x)$ for T_xM. If this basis provides a positive orientation for M, then assign $+1$ to γ and -1 otherwise. From above we know $W(q,p) \cup \{q,p\}$ is compact and 1-dimensional, and also can be acted upon by $\mathbb{R}$ by flowing for time $t \in \mathbb{R}$. Therefore $\mathcal{M}(q,p) = W(q,p)/\mathbb{R}$ is a compact 0-manifold, hence by previous work consists of finitely many points, the number of which is exactly the number of flows $\gamma : q \to p$. Because we label each such flow with a ± 1 depending on orientation, we can define $n(q,p) \in \mathbb{Z}$ as the sum of the labels.

There is a second method of defining $n(q,p)$ which hearkens back to oriented intersection numbers. Given the ties to our previous work, we state this definition here.

Definition 34 Let $c \in (f(p), f(q))$ be a regular value and let $S^u(q) = W^u(q) \cap f^{-1}(c)$, $S^s(p) = W^s(p) \cap f^{-1}(c) \subseteq f^{-1}(c)$. Then $S(q,p) = S^u(q) \cap S^s(p)$ is a (finite order) 0-dimensional manifold, whose intersection number we define to be $n(q,p) \in \mathbb{Z}$.

With this number in mind we are finally able to construct the desired chain complex. The same hypotheses associated with M still apply.

Definition 35 Let $f : M \to \mathbb{R}$ be Morse-Smale and assume orientations for unstable manifolds associated with f have been chosen. Let $C_k(f)$ be the free abelian group of index k critical points of f and define $C_*(f) = \oplus_{k=0}^{m} C_k(f)$ where $m = \dim M$. The Morse-Smale-Witten boundary operator (abbreviated MSW) is a homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ given by

$$\partial_k(q) = \sum_{p \in Cr_{k-1}(f)} n(q,p)p.$$

The pair $(C_*(f), \partial_*)$ is the MSW chain complex of f.

8.4 Morse Homology Theorem

Theorem 21 The homology of the MSW chain complex $(C_*(f), \partial_*)$ is isomorphic to the singular homology $H_*(M, \mathbb{Z})$.

Remark 1 It is fairly intuitive to see the relation between the numbers $n(q,p)$ and the coefficient ring $\mathbb{Z}$ of $H_*(M, \mathbb{Z})$. However, for greater generality, the MSW chain complex can be constructed with coefficients in any commutative ring R via the tensor product $C_k(f) \otimes R$ and an application of the Universal Coefficient Theorem such that $(C_*(f), \partial_*) \cong H_*(M, R)$.

To motivate the Theorem, recall in Theorems 16 and 17 that we used a Morse function $f : M \to \mathbb{R}$ to prove any manifold M of the specific type we are studying is homotopy equivalent to a CW complex X. Well known in algebraic topology is that the CW homology of X, denoted by $(\underline{C}_*(X), \underline{\partial}_*)$, where the free group $\underline{C}_k(X)$ is generated by the k-cells of X and the boundary operator is induced by the homology exact sequence of $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$. Assuming now that $f : M \to \mathbb{R}$ satisfies Morse-Smale transversality under the metric g, we can identify $C_k(f)$ and $\underline{C}_k(X)$ since they are free abelian groups with generators indexed by critical points of index k. Thus to prove Theorem 21 it is enough to show that

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{k+2}} & \underline{C}_{k+1}(X) & \xrightarrow{\partial_{k+1}} & \underline{C}_k(X) & \xrightarrow{\partial_k} & \underline{C}_{k-1}(X) & \xrightarrow{\partial_{k-1}} & \cdots \\
 & & \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
\cdots & \xrightarrow{\partial_{k+2}} & C_{k+1}(f) & \xrightarrow{\partial_{k+1}} & C_k(f) & \xrightarrow{\partial_k} & C_{k-1}(f) & \xrightarrow{\partial_{k-1}} & \cdots
\end{array}$$

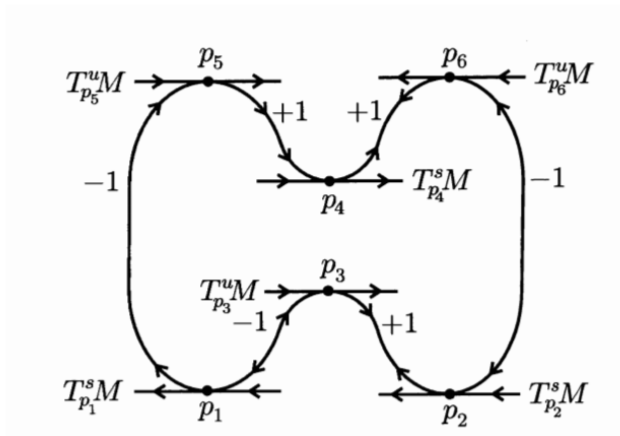
is a commutative diagram, as then $H_*(C_*(f), \partial_*) \cong H_*(\underline{C}_*(X), \underline{\partial}_*) \cong H_*(X, \mathbb{Z}) \cong H_*(M, \mathbb{Z})$. Unfortunately, to rigorously prove this we need to compare the two boundary operators using something known as the Conley index. Rather than do this here, we compute the MSW homology of several standard manifolds to verify that at least in these cases the desired isomorphism holds.

Example 1: S^1. Orient the circle clockwise, let $f : M \to \mathbb{R}$ be the height function, and let the top of the circle be N, similarly S for the bottom. Orient $T_N(S^1)$ from left to right, and orient $T_S(S^1)$ from right to left. We then see points x in the eastern hemisphere are such that $-(\nabla f)(x)$ agrees with the orientation of $T_x W^u(N)$, whereas points x in the western hemisphere have $T_x W^u(N)$ oriented in an opposing fashion to $-(\nabla f)(x)$. Thus by our previous developments concerning $n(p,q)$ we see bases on the LHS assigned a -1 while bases on the RHS are assigned a $+1$. As a result we find

$$\begin{array}{ccccc}
C_1(f) & \xrightarrow{\partial_1 = 0} & C_0(f) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\langle N \rangle & \xrightarrow{\partial_1 = 0} & \langle S \rangle & \longrightarrow & 0
\end{array}.$$

From this we can compute directly $H_k(C_*(f), \partial_*) = ker(\partial_k)/im(\partial_{k+1})$ to find $H_k(C_*(f), \partial_*) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 1 \\ 0, & \text{else} \end{cases}$.

Example 1a: Deformed S^1. Since homology should remain invariant under such transformations this is a valuable test. Again our function $f : S^1 \to \mathbb{R}$ is the height function but now we have many critical points p_i, $i = 1, \dots, 6$. The image below provides all the orientation assignments that have been chosen



Following the same methods as when the circle was not deformed, we find the MSW chain complex looks like

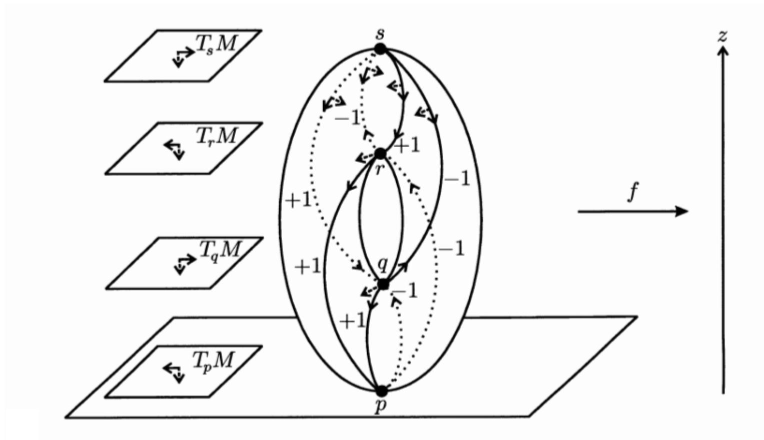
$$\begin{array}{ccccc} C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \langle p_3, p_5, p_6 \rangle & \xrightarrow{\partial_1} & \langle p_1, p_2, p_4 \rangle & \longrightarrow & 0 \end{array}.$$

Now though, the boundary operator is a bit more difficult to describe, as we must compute $n(p_i, p_j)$ for pairs i, j ranging over $1, \dots, 6$. These coefficients are arranged in the following matrix (a_{ij}) where $a_{ij} = n(p_i, p_j)$:

$$(a_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We then can calculate $H_1(C_*(f), \partial_*) = ker(\partial_1)/0 = ker(\partial_1)$. Using the definition of the boundary operator, we know $\alpha = ap_3 + bp_5 + cp_6 \in ker(\partial_1)$ provided $\partial_1(\alpha) = 0$, i.e. $a\partial_1(p_3) + b\partial_1(p_5) + c\partial_1(p_6) = a(p_2 - p_1) + b(p_4 - p_1) + c(p_4 - p_2) = 0$. We see then if $a = -b = c$ that the equation is satisfied, hence $H_1(C_*(f), \partial_*) = ker(\partial_1) = \langle p_3 - p_5 + p_6 \rangle \cong \mathbb{Z}$. Similarly, $H_0(C_*(f), \partial_*) = ker(\partial_0)/im(\partial_1) = \langle p_1, p_2, p_4 \rangle / \langle p_2 - p_1, p_4 - p_1, p_4 - p_1 \rangle = \{(p_1, p_2, p_4) \,|\, p_1 = p_2 = p_4 \in \mathbb{Z}\} \cong \mathbb{Z}$. It is clear the homology groups for $k \neq 0, 1$ are 0, hence we arrive again at the conclusion $H_k(C_*(f), \partial_*) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 1 \\ 0, & \text{else} \end{cases}$ as desired.

Example 2: Tilted T^2. Rather surprisingly, this example is straightforward. In the image below, the orientation chosen for T^2 at a point is indicated by the following ordering of tangent vectors: solid, then dashed.



As seen prior, there are four critical points p, q, r, s of index $0, 1, 1, 2$ respectively. The resulting chain complex looks like

$$\begin{array}{ccccccc}
C_2(f) & \xrightarrow{\partial_1} & C_1(f) & \longrightarrow & C_0(f) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\langle s \rangle & \xrightarrow{\partial_2 = 0} & \langle q, r \rangle & \xrightarrow{\partial_1 = 0} & \langle p \rangle & \longrightarrow & 0
\end{array}.$$

It is then a standard computation to determine $H_k(C_*(f), \partial_*) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$ as expected.

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