KNEADING INVARIANTS OF FIBONACCI QUOTIENT ROTATIONS

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Abstract. This paper studies the sequence of rotations of the unit circle by angles of quotients of adjacent Fibonacci numbers. An equivalence relation is placed on the unit circle that converts it into an interval in the real line. On this interval, each Fibonacci quotient rotation forms a real, quadratic, one dimensional dynamical system. We give an inductive construction for the kneading invariants of this sequence of dynamical systems.

1. Introduction

We consider the quadratic family of polynomials, $P_c\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $P_c(z) = z^2 + c$, for $c \in \mathbb{C}$. Given $c \in \mathbb{C}$, we define the *filled Julia set*, K_c as those $z \in \mathbb{C}$ such that the orbit of z by P_c, $\{P_c^n(z)\colon n \in \mathbb{N}\}$, is bounded and *Julia set* as the boundary of the filled Julia set, ∂K_c.

The *Mandelbrot set*, M, is the set

$$M = \{c \in \mathbb{C}\colon K_c \text{ is connected}\}.$$

It is known that M is compact and connected [3]. By the Riemann mapping theorem, there is an analytic homeomorphism $\phi_M\colon \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{D}$ (where D is the unit disk). Define the *external ray* of argument $t \in \mathbb{R}/\mathbb{Z}$ to be

$$R(t) = \phi_M^{-1}\left(\left\{\rho e^{2i\pi t}\right\}_{\rho>1}\right).$$

If $\phi_M^{-1}(\rho e^{2i\pi t})$ has limit $x \in M$ as $\rho \to 1$ then we say $R(t)$ lands at x. In this case x has *external argument* t. Given $t \in \mathbb{R}/\mathbb{Z}$ we say it lands if it is the external argument of some $x \in M$ [4]. We are interested in which rays land and where. It is known that all external rays of rational argument land [4].

To understand the proof, we first need the following definitions. We consider the following open subset of M:

$$M' = \{c \in \mathbb{C}\colon P_c \text{ has a finite attracting cycle}\}.$$

Let W be a component of M'. There exists a conformal isomorphism ρ_W that maps W to the unit disc, D. We call the unique point c_W the *center of* W if $\rho_W(c_W) = 0$. Using ρ_W extended to the ∂W and ∂D we can define the internal arguments on ∂W as follows. If $c \in \partial W$ has that $\rho_W(c) = \exp(2\pi\gamma)$ then γ is the *internal argument* of c. The point in ∂W with internal argument 0 is called the root of W. This was first considered in [3] but for a more accessible presentation (which this presentation is based on) see [7].

For $\theta \in \mathbb{R}/\mathbb{Z}$ define

$$T(\theta) = \frac{1}{2} + \frac{\theta}{4}.$$

One of the main results of [1] is the following theorem.

Theorem 1 ([1])**.** *Let $c \in \partial W_0$ be a parameter with rational internal argument γ and external arguments θ^-, θ^+ with $0 < \theta^- < \theta^+ < 1/3$. Then*

1. *the external ray $R_M(T(\theta^-))$ lands at c_1 which is a real Misiurewicz parameter (i.e., c_1 is a pre-periodic point), and*
2. *the external ray $R_M(T(\theta^+))$ lands at the real parameter c_2 which is the root of a primitive hyperbolic component.*

The proof makes use of a Hubbard tree. A *tree* is an finite, connected, and acyclic graph embedded in $\mathbb{C}$. A *Hubbard tree* of a given map P_c is a tree, $T \subset K_c$, such that T contains the orbit of 0 and no subtree T' both contains the orbit of 0 and has that $P_c(T') \subset T$. Hubbard trees distill all the combinatorial information of the map they represent into a simple structure. We can generalize Hubbard trees with the following definition. An *abstract Hubbard Tree* is a tree with a continuous and onto map g such that: g is at most two to one, except for a single point, called the *critical point*, g is a local homeomorphism, and every endpoint of T lies on the forward orbit of the critical point [2]. An abstract Hubbard Tree is said to be expanding if for each edge with endpoints v_1, v_2 there is a $n \in \mathbb{N}$ such that the number of edges between $g^n(v_1)$ and $g^n(v_2)$ is strictly larger than 1 [1]. Abstract Hubbard Trees and can be realized as Hubbard tree with the following Theorem.

Theorem 2 ([1])**.** *Any abstract Hubbard tree H can be realized as a tree associated with a polynomial P with a finite critical orbit if and only if H is expanding.*

We now discuss the proof of Theorem 1. Note that if $c \in \partial W_0 \setminus \{1/4\}$ with a rational internal argument, then c is the root of another component. Fix $c \in \partial W_0$ and let W be the component of which c is the root. Let c_0 be the center of W. The proof of Theorem 1 constructs an abstract Hubbard Tree and then uses Theorem 2 to obtain a map $P_{c_0'}$ where c_0' is a real center of a component of M, W'. Theorem 1 is then proved by showing that $T(\theta)$ lands at the root of W' [1]. The abstract Hubbard Tree that is constructed is the object of study of this paper and is covered in detail in Section 3.

Let f_n be the map attained from using the fraction F_n/F_{n+1} in the construction. This function will have a finite kneading invariant, denoted w^n. Split w^n into three pieces:

$$w^n = \Delta_1^n \Delta_2^n \Delta_3^n$$

where Δ_1^n and Δ_3^n have length F_{n-1} and Δ_2^n has length F_{n-2}. Let $\overline{\Delta_1^n}$ be $w_1^n \overline{w_2^n} \dots w_{F_{n-1}}^n$. This paper proves the following:

Theorem 3 (Inductive construction of the kneading invariant)**.** *Suppose $n \in \mathbb{N}$ with $n > 2$. Let f_n be the map from the Fibonacci quotient F_n/F_{n+1} and let w^n be the periodic component of its kneading invariant. Then we have*

$$w^n = \begin{cases} w^{n-1}\overline{\Delta_1^{n-1}}\Delta_2^{n-1} & n \text{ odd} \\ w^{n-1}\Delta_1^{n-1}\Delta_2^{n-1} & n \text{ even} \end{cases}$$

2. Background

This section provides the background information necessary to understand this paper. Throughout this paper $\mathbb{N} = \{1, 2, \dots\}$ is the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers. For a finite set S, we define $|S|$ to be the number of elements of S. If $x \in \mathbb{R}$ we denote by $|x|$ the absolute value of x. Define $[a, b] = [\min(a, b), \max(a, b)]$ and $(a, b) = (\min(a, b), \max(a, b))$. Throughout this paper by $x \pmod y$ we mean the unique $r \in \mathbb{N}$ with $0 \le r < n$ such that r is the remainder of x divided by y. We say $\mathbb{R}/\mathbb{Z}$ is the set of equivalence classes of $\mathbb{R}$ where $x \sim y$ if $x - y \in \mathbb{Z}$. Each class has an element in $[0, 1)$ which we is the representative for that class. By $x \pmod 1$ we mean the representative of the class containing x.

2.1. **Dynamical Systems.**

A real, one-dimensional dynamical systems is a pair (f, I) where $I \subset \mathbb{R}$ is a compact interval and $f \colon I \to I$ is a continuous function. For $n \in \mathbb{N}_0$ let f^n denote the n fold composition of f with itself, i.e., f^0 is the identity and $f^n = f(f^{n-1})$ for $n > 0$. For every point x in I, we define *the orbit of x by f*, by

$$O_f(x) = \{x, f(x), f^2(x), \dots\}.$$

The goal of dynamical systems is to describe how the orbits are distributed. If there is $n \in \mathbb{N}$ such that $f^n(x) = x$ then we say that x is *n-periodic* and $O_f(x)$ is a periodic orbit.

A continuous map f defined on an interval $I = [a, b]$ is called *unimodal* if there is $c \in (a, b)$ such that f is strictly monotone on $[a, c)$ and $(c, b]$ and c is a global maximum or minimum. The point c is called the *turning point of f*. For an unimodal map f with turning point c, write $c_i = f^i(c)$ [2].

2.2. **Kneading Theory.** Given a dynamical system (f, I) where f is an unimodal map and a point $x \in I$ we define the *itinerary of f and x*, $I(f, x)$, to be the sequence $e_1, e_2, \dots$ where

$$e_i = \begin{cases} 0 & f^i(x) < c \\ 1 & f^i(x) > c \\ * & f^i(x) = c \end{cases}$$

If the last case occurs, the itinerary is finite and stops before the star. In other words if x is n-periodic then $I(x, f)$ is a finite sequence of $n-1$ binary values. We call $I(c_1, f)$ the *kneading sequence* or the *kneading invariant* of f [2].

2.3. **Diophantine Approximations.** This subsection provides the necessary details on Diophantine approximations. The reference for this information is Section 2 of Chapter 1 of [6]. Let $\alpha \in \mathbb{R}$ be irrational and set $a_0 = \lfloor \alpha \rfloor$ and $1/\alpha_1 = \alpha - a_0$. Inductively for all $n \in \mathbb{N}$ we define $a_n = \lfloor \alpha_n \rfloor$ and $1/\alpha_{n+1} = \alpha_n - a_n$. This defines the *continued fraction* expansion of α, written $[a_0, a_1, \dots]$. In other words we have that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

Let $n \in \mathbb{N}_0$, then we have that $[a_0, a_1, \dots, a_n]$ is a rational number. Define the *n-th principal convergent*, p_n/q_n, to be $[a_0, a_1, \dots, a_n]$ where $p_n, q_n \in \mathbb{Z}$. We note that $p_0 = a_0$, $q_0 = 1$, $p_1 = p_0 a_1 + 1$, and $q_1 = a_1$. The following theorem from [6] gives a formula for p_n and q_n when $n \geq 2$.

Theorem 4. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then*

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}.$$

We will need the following equations concerning principal convergents $p_n/q_n \to \phi$. For their proofs see [6]. First

$$q_{n+1}\alpha - p_{n+1} = \frac{(-1)^{n+1}}{\alpha_{n+2} q_{n+1} + q_n}. \tag{1}$$

And second

$$q_n \alpha - p_n = \frac{(-1)^n \alpha_{n+2}}{\alpha_{n+2} q_{n+1} + q_n}. \tag{2}$$

Lastly we need the following Theorem from [6].

Theorem 5. *For n even the n-th principal convergents form a strictly increasing sequence. For n odd the n-th principal convergents form a strictly decreasing sequence.*

We now turn our attention to the special case where $\alpha = \phi$, where ϕ is defined as the positive solution to the equation

$$x^2 - x - 1 = 0.$$

By rewriting we get

$$\phi = 1 + \frac{1}{\phi}. \tag{3}$$

Thus for all $n \in \mathbb{N}_0$, $\phi_n = \phi$ and $a_n = \lfloor \phi \rfloor = 1$. Hence $p_1 = 1$, $p_2 = 2$ and in general $p_n = p_{n-1} + p_{n-2}$. In particular $p_n = F_{n+1}$. Similarly $q_n = F_n$ for all $n \in \mathbb{N}_0$. Plugging this into 1 and 2 we get the following equations

$$F_{n+1}\phi - F_{n+2} = \frac{(-1)^{n+1}}{\phi F_{n+1} + F_n} \tag{4}$$

$$F_n\phi - F_{n+1} = \frac{(-1)^n \phi}{\phi F_{n+1} + F_n}. \tag{5}$$

We will now prove a stronger version of Theorem 6 in Chapter 1, Section 2 of [6] for the specific case of ϕ. Our proof follows the same technique as the proof in [6] but changes some inequalities. First, we make the following change to the definition of a best approximation appearing in [6]. We say that a/b with $a, b \in \mathbb{N}$ is a best approximation to ϕ if for all $b' \in \mathbb{N}$ with $1 \leq b' < b$,

$$\|b\phi\| = |b\phi - a| \quad \text{and} \quad \frac{\|b'\phi\|}{b'} > \frac{\|b\phi\|}{b} \tag{6}$$

Theorem 6 ([6]). *The best approximations to ϕ are its principal convergents. In particular, for $n \in \mathbb{N}$, F_n is the smallest integer $q > F_{n-1}$ such that $\|q\alpha\|/q < \|q_n\alpha\|/q_n$.*

Proof. First we will show that if $a/b \in \mathbb{Q}$ is a best approximation with $a, b \in \mathbb{N}$ then there is $n \in \mathbb{N}$ such that $a = F_{n+1}$ and $b = F_n$. Then we will show that F_{n+1}/F_n is a best approximation. Suppose that a/b is a best approximation. We can assume that $a/b \geq 1 = F_2/F_1$ as otherwise

$$\frac{\|\phi\|}{1} = \left|\phi - \frac{F_1}{F_0}\right| = |\phi - 1| < \left|\phi - \frac{a}{b}\right| = \frac{\|b\phi\|}{b}$$

contradicting that a/b is a best approximation as $1 < b$ yet

$$\frac{\|\phi\|}{1} < \frac{\|b\phi\|}{b}.$$

Similarly we can rule out that $a/b > 2 = F_3/F_2$. Thus we can assume that there is a $n > 2$ such that a/b is between F_n/F_{n-1} and F_{n+2}/F_{n+1} using Theorem 5. Notice

$$\frac{1}{bF_{n-1}} \leq \left|\frac{a}{b} - \frac{F_n}{F_{n-1}}\right| < \left|\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}}\right| = \frac{1}{F_nF_{n-1}}$$

and so $b > F_n$. The same argument shows that $b < F_{n+1}$. We wish to show that

$$\frac{\|b\phi\|}{b} > \frac{\|F_n\phi\|}{b}.$$

Rewriting this is the same as

$$\left|\phi - \frac{a}{b}\right| > \left|\phi - \frac{F_{n+1}}{F_n}\right|.$$

Note that

$$\left|\phi - \frac{a}{b}\right| = \left|\frac{a}{b} - \frac{F_{n+2}}{F_{n+1}}\right| + \left|\phi - \frac{F_{n+2}}{F_{n+1}}\right| \geq \frac{1}{bF_{n+1}} + \left|\phi - \frac{F_{n+2}}{F_{n+1}}\right| \geq \frac{1}{F_{n+1}^2} + \left|\phi - \frac{F_{n+2}}{F_{n+1}}\right|$$

Hence it suffices to show that

$$\frac{1}{F_{n+1}^2} + \left|\phi - \frac{F_{n+1}}{F_n}\right| > \left|\phi - \frac{F_{n+2}}{F_{n+1}}\right|$$

or that

$$\frac{1}{F_{n+1}^2} > \left|\phi - \frac{F_{n+1}}{F_n}\right| - \left|\phi - \frac{F_{n+2}}{F_{n+1}}\right|.$$

Using 4 and 5 we get

$$\begin{aligned}
\frac{1}{F_{n+1}^2} &> \frac{\phi}{F_n(\phi F_{n+1} + F_n)} - \frac{1}{F_{n+1}(\phi F_{n+1} + F_n)} \\
\frac{1}{F_{n+1}^2} &> \frac{1}{\phi F_{n+1} + F_n} \frac{\phi F_{n+1} - F_n}{F_{n+1}F_n} \\
\frac{F_n}{F_{n+1}} &> \frac{\phi F_{n+1} - F_n}{\phi F_{n+1} + F_n} \\
F_n(\phi F_{n+1} + F_n) &> F_{n+1}(\phi F_{n+1} - F_n) \\
F_n(F_{n+1} + F_n) &> \phi F_{n+1}(F_{n+1} - F_n) \\
\frac{F_n F_{n+2}}{F_{n+1}F_{n-1}} &> \phi.
\end{aligned}$$

Instead we will prove that LHS is larger than $2 > \phi$. We get

$$\begin{aligned}
2F_{n+1}F_{n-1} &< F_n F_{n+2} \\
2F_{n+1}F_{n-1} &< F_n F_{n+1} + F_n^2 \\
2F_{n+1}F_{n-1} &< F_n F_{n+1} + F_{n+1}F_{n-1} + (-1)^n \\
F_{n+1}F_{n-1} &< F_n F_{n+1} + (-1)^n \\
F_{n+1}F_{n-1} &< F_{n-1}F_{n+1} + F_{n-2}F_{n+1} + (-1)^n \\
(-1)^{n+1} &< F_{n-2}F_{n+1}.
\end{aligned}$$

This is true for all $n > 2$. Thus we have proved that

$$\left|\phi - \frac{a}{b}\right| < \left|\phi - \frac{F_{n+1}}{F_n}\right|$$

and in particular there is a $F_n = b' < b$ such that

$$\frac{\|b'\phi\|}{b'} > \frac{\|b\phi\|}{b}.$$

Contradicting that a/b is a best approximation. Thus we know that if a/b satisfies 6 there is n such that $a/b = F_{n+1}/F_n$. We now show that for all $n \in \mathbb{N}$, F_{n+1}/F_n is a best approximation. This part of the proof does not change from the proof in [6] but is still included for completeness. We proceed with induction. For $n = 1$, the property vacuously holds. Suppose that F_{n+1}/F_n is a best approximation. Let b be the smallest integer such that

$$\frac{\|b\phi\|}{b} < \frac{\|F_n\phi\|}{F_n}$$

and a be such that $\|b\phi\| = |b\phi - a|$. Since we know that F_{n+1}/F_n meets property 6 we must also have that a/b meets property 6 and hence $b = F_m$ for some m. But we have that $\|F_n\phi\| > \|F_{n+1}\phi\|$ so $b = F_{n+1}$. Hence, $a = F_{n+2}$ proving the claim. □

Lastly, we make the following remark.

Lemma 1. *Let $k \in \mathbb{N}$, then*

$$\left\|\frac{k}{\phi}\right\| = \|k\phi\|$$

Proof.

$$\left\|\frac{k}{\phi}\right\| = \|k(\phi-1)\| = \|k\phi - k\| = \|k\phi\|$$

□

2.4. **Fibonacci Facts.** In this section, we state some facts about Fibonacci numbers that will play an important role in this paper.

Theorem 7. *Let $p, q \in \mathbb{N}$ with $p > q$. Then*

$$F_pF_{q+1} - F_{p+1}F_q = (-1)^pF_{p-q}.$$

For a proof of Theorem 7 see [5].

Lemma 2. *Let $n \in \mathbb{N}$ then*

$$[F_n\phi] = F_{n+1}.$$

3. Construction

In this section, we describe the sequence of functions of interest. This construction appears in [1] for general fractions $p/q \in \mathbb{Q}$ but for this paper we restrict our attention to those fractions of the form F_n/F_{n+1}. First, we define

$$S^1 = \{e^{2\pi i\theta} \colon \theta \in \mathbb{R}/\mathbb{Z}\} \subset \mathbb{C}.$$

For $n \in \mathbb{N}$ with $n > 2$ we define $R_n \colon S^1 \to S^1$ to be

$$R_n\left(e^{2\pi i\theta}\right) = e^{2\pi i(\theta + F_n/F_{n+1})}$$

for $\theta \in \mathbb{R}/\mathbb{Z}$ ($\theta + F_n/F_{n+1}$ is computed mod 1). In other words, R_n is a rotation of the unit circle by F_n/F_{n+1} turns counterclockwise. Next for $k \in \mathbb{N}_0$ with $k < F_{n+1}$ define

$$z_k^n = R_n^k(1) = e^{2\pi ikF_n/F_{n+1}}.$$

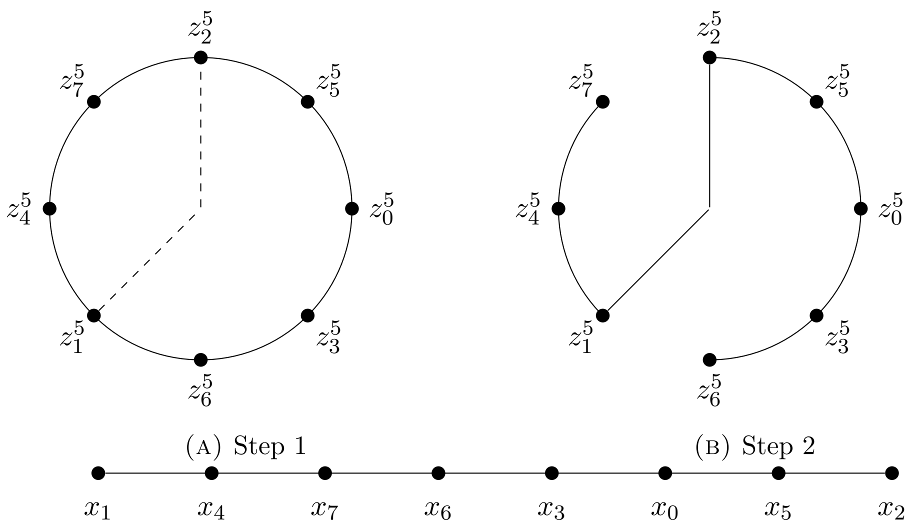
Next, we construct the Hubbard tree. Consider the closed arc in S^1 joining x_1 to x_2 which does not contain 1, B, and the closure of its complement in S^1, B'. To avoid confusion, add a quote to the labeled elements of B'. We map both B unto a real interval such that $x_1 < x_2$ where x_j is the point corresponding to z_j^n. Similarly, map the B' unto a real interval such that $x_1' < x_2'$ where x_j' is the point corresponding to z_j^n. Paste these two intervals together with the relation $x_2 \sim {x_1}'$ and name their common point $\hat{\alpha}$. This process is shown in Figure 1 with $n = 5$ [1].

We define the function, f_n, of the form $x^2 + c$ for $c \in \mathbb{R}$ on this interval by f_n maps $[x_i^n, x_j^n]$ to $[x_{i+1}^n, x_{j+1}^n]$ where the addition is mod F_{n+1}. The existence of such function is guaranteed by Theorem 2 [1]. We can obtain the parameter c by choosing $x_0 = 0$ and then using the fact that $f^{F_{n+1}}(x_0) = x_0$. Note that $f^{F_{n+1}}(0)$ is a degree $2^{F_{n+1}}$ polynomial in the parameter c. Finding the real roots of this polynomial gives us the values of c for which $f^m(0) = 0$. By looking at the orbit of the critical point for each of these maps we can find one that has the constructed orbit for f_n. As an example, this process was completed for $n = 4$ and is shown in Figure 2.

In this paper, we are interested in which side of x_0^n each x_j^n is on as this determines whether w_j^n is 0 or 1. To do this, we note that x_j^n is to the right of x_0^n only when z_j^n is on the counterclockwise arc from z_0^n to z_2^n. Hence, we wish to determine the order in which they occur counterclockwise starting from $z_0^n = 1$. To do so, for $r \in \mathbb{N}_0$ with $r < F_{n+1}$ we define

$$q_r^n = e^{2\pi ir/F_{n+1}}.$$

Note q_r^n is the r-th point counterclockwise starting from 1 in S^1. We have the following correspondence between z_k^n and q_r^n.



(A) Step 1

(B) Step 2

(C) Step 3

FIGURE 1. Process for constructing the critical orbit for f_5 map with the construction.

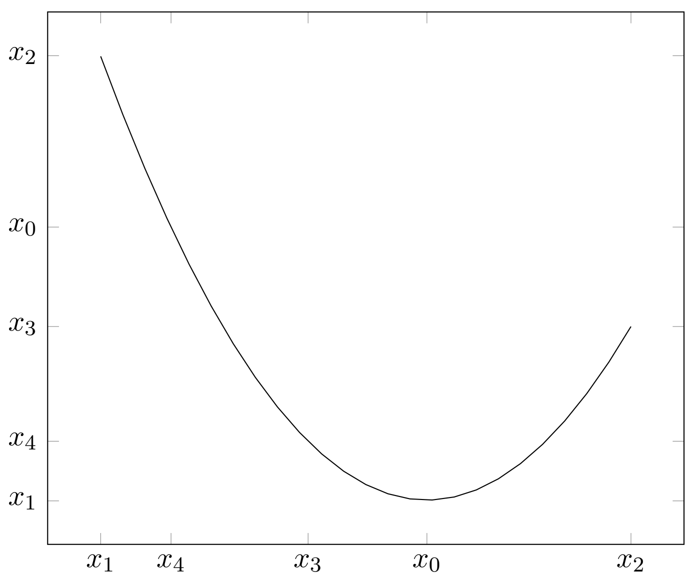


FIGURE 2. The map f_4 obtained by the construction.

Lemma 3. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $n > 2$ and $k < F_{n+1}$. Further $r = kF_n \pmod{F_{n+1}}$ and z_k^n and q_r^n are as defined previously then*

$$z_k^n = q_r^n$$

Proof. Fix $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ with $k < F_{n+1}$. This lemma is the same as proving

$$\frac{kF_n}{F_{n+1}} \pmod 1 = \frac{r}{F_{n+1}}$$

Write

$$kF_n = mF_{n+1} + r.$$

Then note

$$\frac{kF_n}{F_{n+1}} \pmod 1 = \frac{r}{F_{n+1}}.$$

This concludes the proof. □

We will now state a few observations about the ordered set Z_n for $n \in \mathbb{N}$ with $n > 2$ that follow from Lemma 3 and from d'Ocagne's identity.

Remark 1.

(1) $z_0^n = q_0^n$
(2) $q_1^n = z_{F_n}^n$ if n is odd and $q_1^n = z_{F_{n-1}}^n$ if n is even
(3) $z_2^n = q_{F_{n-2}}^n$
(4) If z_k^n is in the counterclockwise arc of S^1 from z_0^n to z_2^n then $w_k^n = 1$ otherwise $w_k^n = 0$.

Using these observations we get that $w_k^n = 1$ if and only if

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2}. \tag{7}$$

Similarly $w_k^n = 0$ if and only if

$$F_{n-2} < kF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}. \tag{8}$$

For $n \in \mathbb{N}$ with $n > 2$ and $k \in \mathbb{N}_0$ with $k < F_{n+1}$, let $I_n(k)$ be such that $z_k^n = q_{I_n(k)}^n$. By Lemma 3 we know

$$I_n(k) = kF_n \pmod{F_{n+1}}.$$

Lastly in this section we remark that the indexing function is a bijection.

Lemma 4. *Let $n \in \mathbb{N}$ with $n > 2$ then*

$$I_n\colon \{0, \ldots, F_{n+1} - 1\} \to \{0, \ldots, F_{n+1} - 1\}$$

is a bijection.

Proof. Let $n \in \mathbb{N}$ with $n > 2$. Note that $\gcd(F_n, F_{n+1}) = 1$ and so we know that F_n has an inverse, F_n^{-1} in $\mathbb{Z}_{F_{n+1}}$. Thus we can define J_n with the same domain and range as I_n by $J(k) = kF_n^{-1} \pmod{F_{n+1}}$ as an inverse to I_n. Notice for $k \in \mathbb{N}_0$ with $k < F_{n+1}$.

$$\begin{aligned} I_n(J_n(k)) &= (kF_n \pmod{F_{n+1}})F_n^{-1} \pmod{F_{n+1}} \\ &= k \pmod{F_{n+1}} \\ &= k \end{aligned}$$

where the last line follows from $k < F_{n+1}$. Similarly

$$\begin{aligned} J_n(I_n(k)) &= (kF_n^{-1} \pmod{F_{n+1}})F_n \pmod{F_{n+1}} \\ &= k \pmod{F_{n+1}} \\ &= k. \end{aligned}$$

Hence we know that J_n is I_n^{-1} and so I_n must be a bijection. □

The remainder of this paper gives an inductive construction for the kneading invariants of f_n for $n > 2$. Note that the critical orbit for each f_n is finite and hence has a finite kneading invariant of length $F_{n+1} - 1$, which we denote w^n. The j-th element of w^n is denoted as w_j^n. Lastly we will extend w^n to have length F_{n+1}. We define

$$w_{F_{n+1}}^n = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

4. Results

4.1. **Main Result.** Split w^n in the following way:

$$w^n = \Delta_1^n \Delta_2^n \Delta_3^n.$$

Where Δ_1^n and Δ_3^n are of length F_{n-1} and Δ_2^n is of length F_{n-2}. For all $n, j \in \mathbb{N}$ with $j < 3$ and $n > 2$, if $\Delta_j^n = e_1, \ldots, e_s$ where s is either F_{n-1} or F_{n-2} define $\overline{\Delta_j^n}$ to be $e_1, \overline{e_2}, \ldots, e_s$. If the length of Δ_j^n is smaller than 2 then $\overline{\Delta_j^n} = \Delta_j^n$. Recall if $x \in \{0,1\}$ $\overline{x}$ is $1-x$. The main result of this paper is

Theorem 3 (Inductive construction of the kneading invariant)**.** *Suppose $n \in \mathbb{N}$ with $n > 2$ then*

$$w^n = \begin{cases} w^{n-1}\overline{\Delta_1^{n-1}}\Delta_2^{n-1} & n \text{ odd} \\ w^{n-1}\Delta_1^{n-1}\Delta_2^{n-1} & n \text{ even} \end{cases}$$

To prove Theorem 3 we will prove this equivalent lemma.

Lemma 5. *For all $n > 2$*

$$w^n = \begin{cases} \Delta_1^{n-1}\Delta_2^{n-1}\Delta_1^{n-1}\overline{\Delta_1^{n-1}}\Delta_2^{n-1} & n \text{ odd} \\ \Delta_1^{n-1}\Delta_2^{n-1}\overline{\Delta_1^{n-1}}\Delta_1^{n-1}\Delta_2^{n-1} & n \text{ even} \end{cases}$$

Proof. (Lemma 3 implies Theorem 3)
We will use induction on n. See Figure 3 for the base case. Now suppose $n > 2$ and

$$w^n = \begin{cases} w^{n-1}\overline{\Delta_1^{n-1}}\Delta_2^{n-1} & n \text{ odd} \\ w^{n-1}\Delta_1^{n-1}\Delta_2^{n-1} & n \text{ even} \end{cases}. \tag{9}$$

By Lemma 5 we have

$$w^{n+1} = \begin{cases} \Delta_1^n\Delta_2^n\Delta_1^n\overline{\Delta_1^n}\Delta_2^n & n \text{ even} \\ \Delta_1^n\Delta_2^n\overline{\Delta_1^n}\Delta_1^n\Delta_2^n & n \text{ odd.} \end{cases}$$

Suppose that n is even. Then by (9) we know $\Delta_3^n = \Delta_1^{n-1}\Delta_2^{n-1} = \Delta_1^n$. If n is odd, we have that $\Delta_3^n = \overline{\Delta_1^{n-1}}\Delta_2^{n-1} = \overline{\Delta_1^n}$. Hence

$$w^{n+1} = \begin{cases} w^n\overline{\Delta_1^n}\Delta_2^n & n \text{ odd} \\ w^n\Delta_1^n\Delta_2^n & n \text{ even} \end{cases}.$$

This concludes the proof. □

4.2. **Examples.** We will walk through how to use this pattern. Looking at Figure 3 we see that

$$w^3 = 010$$

and

$$\Delta_1^3 = 0 \qquad \Delta_2^3 = 1 \qquad \Delta_3^3 = 0.$$

Using Lemma 3 we see that

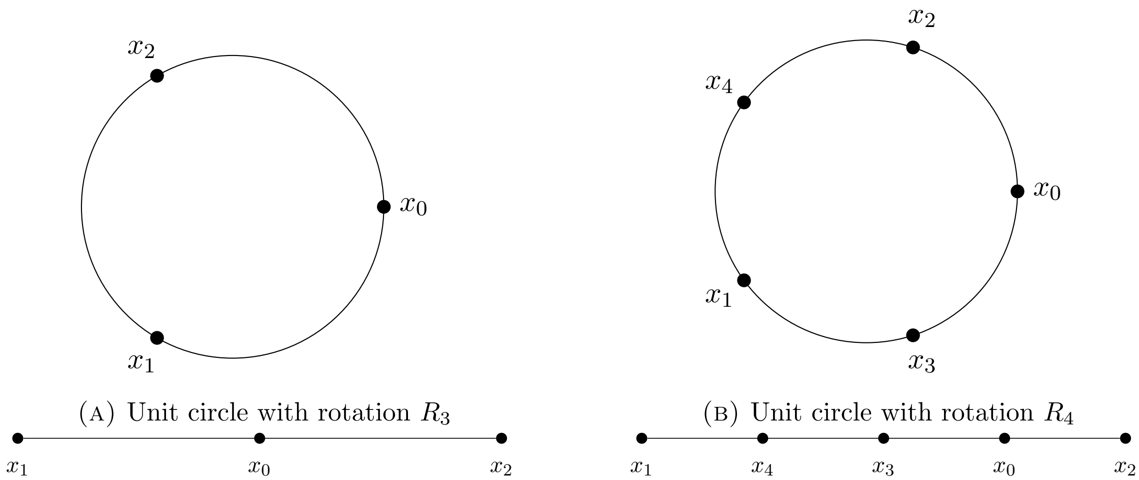
$$\begin{aligned} w^4 &= \Delta_1^3\Delta_2^3\overline{\Delta_1^3}\Delta_1^3\Delta_2^3 \\ &= 01001. \end{aligned}$$

Now we have that

$$\Delta_1^4 = 01 \qquad \Delta_2^4 = 0 \qquad \Delta_3^4 = 01.$$

This time by Theorem 3,

$$\begin{aligned} w^5 &= w^4\overline{\Delta_1^4}\Delta_2^4 \\ &= 01001000. \end{aligned}$$



(A) Unit circle with rotation R_3

(B) Unit circle with rotation R_4

(C) Corresponding interval for f_3.

(D) Corresponding interval for f_4.

FIGURE 3. Construction of the critical orbits for f_3 and f_4.

Again

$$\Delta_1^5 = 010 \qquad \Delta_2^5 = 01 \qquad \Delta_3^5 = 000$$

and so

$$\begin{aligned} w^6 &= w^5 \Delta_1^5 \Delta_2^5 \\ &= 0100100001001. \end{aligned}$$

4.3. **Proofs.** We will now proof Lemma 3. We will split the proof into three parts, constructing Δ_1^n, Δ_2^n and finally Δ_3^n in Lemmas 6, 13, and 14 respectively.

4.3.1. *Proof of Lemma 6.* We will prove the following:

Lemma 6. *Let $n \in \mathbb{N}$ with $n > 2$ then*

$$\Delta_1^n = \Delta_1^{n-1} \Delta_2^{n-1}.$$

Recall that for $x \in \mathbb{R}$,

$$\begin{aligned} \lfloor x \rfloor &= \max\{a \in \mathbb{Z} \colon a \leq x\} \\ \lceil x \rceil &= \min\{a \in \mathbb{Z} \colon a \geq x\} \\ [x] &= \begin{cases} \lfloor x \rfloor & x - \lfloor x \rfloor < \lceil x \rceil - x \\ \lceil x \rceil & \textit{else} \end{cases}. \end{aligned}$$

This lemma requires comparing $kF_n \pmod{F_{n+1}}$ with $kF_{n-1} \pmod{F_n}$. The following observation gives us a way to do this. The idea is to note that for a fixed $k \in \mathbb{N}$ and every $n \in \mathbb{N}$ we can write:

$$kF_n = q_n F_{n+1} + r_n.$$

And for all $n \in \mathbb{N}$,

$$q_n = \left\lfloor \frac{kF_n}{F_{n+1}} \right\rfloor$$

and

$$r_n = kF_n \pmod{F_{n+1}}.$$

We know that kF_n/F_{n+1} converges to k/ϕ and the floor function is continuous except at the integers. However, for $k \in \mathbb{N}$, k/ϕ cannot be an integer. Thus q_n converges to $\lfloor k/\phi \rfloor$. Once this occurs then for all subsequent n', q'_n will be a constant. We formalize this with the following lemma. Recall that for $x \in \mathbb{R}$

$$\|x\| = \min\{|x-n| \colon n \in \mathbb{Z}\} = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}.$$

Lemma 7. *For each $k \in \mathbb{N}$ there is a $N_k \in \mathbb{N}_0$ such that for all $n \geq N_k$,*

$$\left\lfloor \frac{kF_n}{F_{n+1}} \right\rfloor = \left\lfloor \frac{k}{\phi} \right\rfloor$$

and

$$N_k = \left\lceil \frac{\log_\phi \left(\frac{\phi^2+1}{\frac{\phi}{k}\left\|\frac{k}{\phi}\right\|} + 1 \right) - 2}{2} \right\rceil.$$

Proof. Fix $k \in \mathbb{N}$, set

$$\epsilon_k = \left\| \frac{k}{\phi} \right\|.$$

Clearly $\epsilon_k > 0$ and so we know there is N_k such that for all $n \geq N_k$,

$$\left| \frac{kF_n}{F_{n+1}} - \frac{k}{\phi} \right| < \epsilon_k. \tag{10}$$

Fix $n \geq N_k$. Note that

$$\frac{kF_n}{F_{n+1}} \in \left(\frac{k}{\phi} - \epsilon_k, \frac{k}{\phi} + \epsilon_k \right)$$

and that

$$\epsilon_k = \min \left(\frac{k}{\phi} - \left\lfloor \frac{k}{\phi} \right\rfloor, \left\lceil \frac{k}{\phi} \right\rceil - \frac{k}{\phi} \right).$$

As a consequence

$$\frac{k}{\phi} - \epsilon_k \geq \left\lfloor \frac{k}{\phi} \right\rfloor$$
$$\frac{k}{\phi} + \epsilon_k \leq \left\lceil \frac{k}{\phi} \right\rceil$$

hence

$$\left\lfloor \frac{k}{\phi} \right\rfloor < \frac{kF_n}{F_{n+1}} < \left\lceil \frac{k}{\phi} \right\rceil$$

and finally

$$\left\lfloor \frac{k}{\phi} \right\rfloor = \left\lfloor \frac{kF_n}{F_{n+1}} \right\rfloor.$$

Now we construct N_k. We will use an epsilon-delta argument to do so. We wish to find n such that

$$\left| \frac{F_n}{F_{n+1}} - \frac{1}{\phi} \right| < \frac{\epsilon_k}{k}. \tag{11}$$

Use the identity that for all $n \in \mathbb{N}$

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

We get

$$
\begin{aligned}
\left|\frac{\phi^n-(-\phi)^{-n}}{\phi^{n+1}-(-\phi)^{-n-1}}-\frac{1}{\phi}\right| &= \left|\frac{\phi^{n+1}-\phi(-\phi)^{-n}-\phi^{n+1}+(-\phi)^{-n-1}}{\phi^{n+2}-\phi(-\phi)^{-n-1}}\right| \\
&= \left|\frac{-\phi(-\phi)^{-n}+(-\phi)^{-n-1}}{\phi^{n+2}-\phi(-\phi)^{-n-1}}\right| \\
&= \left|\frac{(-\phi)^{-n+1}+(-\phi)^{-n-1}}{\phi^{n+2}-\phi(-\phi)^{-n-1}}\right| \\
&=(\phi^2+1)\left|\frac{(-\phi)^{-n-1}}{\phi^{n+2}-\phi(-\phi)^{-n-1}}\right| \\
&=\frac{\phi^2+1}{\phi}\left|\frac{1}{(-1)^{n+1}\phi^{2n+2}-1}\right| \\
&\leq\frac{\phi^2+1}{\phi}\frac{1}{\phi^{2n+2}-1}.
\end{aligned}
$$

Notice

$$\frac{\phi^2+1}{\phi}\frac{1}{\phi^{2n+2}-1}<\frac{\epsilon_k}{k}$$

happens when

$$n \geq \frac{1}{2}\log\left(\frac{\phi^2+1}{\frac{\phi}{k}\epsilon_k}+1\right)-1=N_k.$$

This proves the claim. □

Recall again that for this paper $x \pmod y$ is defined as the remainder of x divided by y. Lemma 7 allows us to prove the following lemma.

Lemma 8. *For each $k\in\mathbb{N}$ if N_k is as defined above, for all $n\in\mathbb{N}$ with $n \geq N_k+2$ we have*

$$kF_n \pmod{F_{n+1}} = kF_{n-1} \pmod{F_n} + kF_{n-2} \pmod{F_{n-1}}.$$

Proof. Fix $k\in\mathbb{N}$ and let $n\in\mathbb{N}$ such that $n\geq N_k+2$. Write

$$
\begin{aligned}
kF_n &= q_nF_{n+1}+r_n \\
kF_{n-1} &= q_{n-1}F_n+r_{n-1} \\
kF_{n-2} &= q_{n-2}F_{n-1}+r_{n-2}
\end{aligned}
$$

where $0\leq r_i<F_{i+1}$ for all i. Note that

$$q_n=\left\lfloor\frac{kF_n}{F_{n+1}}\right\rfloor.$$

By Lemma 7, as $n>N_k+2$ q_n, q_{n-1}, and q_{n-2} will all be equal. Call their common value q. Rewriting the above we see that

$$
\begin{aligned}
r_n &= kF_n-qF_{n+1} \\
r_{n-1} &= kF_{n-1}-qF_n \\
r_{n-2} &= kF_{n-2}-qF_{n-1}
\end{aligned}
$$

Focusing on r_n and using the definition of the Fibonacci numbers we see that

$$
\begin{aligned}
r_n &= kF_{n-1}+kF_{n-2}-qF_n-qF_{n-1} \\
&= (kF_{n-1}-qF_n)+(kF_{n-2}-qF_{n-1}) \\
&= r_{n-1}+r_{n-2}
\end{aligned}
$$

Lastly note that for all i in $\{n-2, n-1, n\}$ we have that

$$r_i = kF_i \pmod{F_{i+1}}.$$

And so we have

$$kF_n \pmod{F_{n+1}} = kF_{n-1} \pmod{F_n} + kF_{n-2} \pmod{F_{n-1}}$$

as desired. □

To prove Lemma 6 we must prove that for $n \in \mathbb{N}$ with $n > 2$ and $k \in \mathbb{N}_0$ with $k < F_{n-1}$

$$w_k^n = w_k^{n-1}.$$

We need Lemma 8 to apply for each $n \in \mathbb{N}$ such that $k < F_{n-1}$. In particular, it suffices to show n' is such that $F_{n'-2} \le k < F_{n'-1}$ then $n' \ge N_k + 2$. Equivalently, if we say n' is such that $F_n \le k < F_{n+1}$ then we need to show $n' \ge N_k - 1$, in fact we show $n' \ge N_k$. We do so in the following lemma.

Lemma 9. *For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $F_n \le k < F_{n+1}$ then $N_k \le n$.*

The proof of this Lemma comes in two parts. We will first show that in the set $\{N_{F_n}, \ldots, N_{F_{n+1}-1}\}$ has N_{F_n} as it maximum value. We then show that $N_{F_n} \le n$.

Lemma 10. *Let $n \in \mathbb{N}$ with $n > 2$ then*

$$N_{F_n} = \max\{N_{F_n}, N_{F_n+1}, \ldots, N_{F_{n+1}-1}\}$$

Proof. Recall

$$N_k = \left\lceil \frac{1}{2} \log_\phi \left(\frac{\phi^2+1}{\frac{\phi}{k}\left\|\frac{k}{\phi}\right\|} + 1 \right) - 1 \right\rceil.$$

As $\log_\phi$ is increasing, this is equivalent to proving that

$$\frac{\|\frac{F_n}{\phi}\|}{F_n} = \min\left\{ \frac{\|\frac{F_n}{\phi}\|}{F_n}, \frac{\|\frac{F_n+1}{\phi}\|}{F_n+1}, \ldots, \frac{|\frac{F_{n+1}-1}{\phi}\|}{F_{n+1}-1} \right\}.$$

By Lemma 1 this is the same as proving

$$\frac{\|F_n\phi\|}{F_n} = \min\left\{ \frac{\|F_n\phi\|}{F_n}, \frac{\|(F_n+1)\phi\|}{F_n+1}, \ldots, \frac{\|(F_{n+1}-1)\phi\|}{F_{n+1}-1} \right\}.$$

This immediately follows from Theorem 6. □

Lastly, to complete the proof of Lemma 9, we need the following.

Lemma 11. *For all m, $N_{F_m} \le m$.*

Proof. Recall that

$$N_{F_m} = \left\lceil \frac{1}{2} \log_\phi \left(\frac{\phi^2+1}{\phi\epsilon_{F_m}} + 1 \right) - 1 \right\rceil$$

and so we wish to prove

$$\frac{\log_\phi\left(\frac{\phi^2+1}{\phi\epsilon_{F_m}} + 1\right) - 2}{2} \le m.$$

Rewriting we get

$$\left| \frac{1}{\phi} - \left\lceil \frac{F_m}{\phi} \right\rfloor \frac{1}{F_m} \right| \ge \frac{\phi^2+1}{\phi(\phi^{2m+2}-1)}.$$

We can use the fact that $[F_n/\phi] = F_{n-1}$ to see that this is the same as

$$\left|\frac{1}{\phi} - \frac{F_{m-1}}{F_m}\right| \geq \frac{\phi^2+1}{\phi(\phi^{2m+2}-1)}.$$

Expanding this out we get

$$\begin{aligned}
\left|\frac{\phi^m - (-\phi)^{-m} - \phi^m + \phi(-\phi)^{-m+1}}{\phi(\phi^m - (-\phi)^{-m})}\right| &\geq \frac{\phi^2+1}{\phi(\phi^{2m+2}-1)} \\
\left|\frac{-(-\phi)^{-m} + \phi(-\phi)^{-m+1}}{\phi(\phi^m - (-\phi)^{-m})}\right| &\geq \frac{\phi^2+1}{\phi(\phi^{2m+2}-1)} \\
\frac{\phi^2+1}{|\phi^{m+1}(\phi^m - (-\phi)^{-m})|} &\geq \frac{\phi^2+1}{\phi(\phi^{2m+2}-1)} \\
\left|\phi^{m+1}(\phi^m - (-\phi)^{-m})\right| &\leq \phi(\phi^{2m+2}-1) \\
\left|\phi(\phi^{2m} + (-1)^{m-1})\right| &\leq \phi(\phi^{2m+2}-1) \\
\phi^{2m}+1 &\leq \phi^{2m+2}-1 \\
2 &\leq \phi^{2m+1}.
\end{aligned}$$

Which is true for all $m \geq 1$. □

We are now ready to prove Lemma 6.

Lemma 6. *Let $n \in \mathbb{N}$ with $n > 2$,*

$$\Delta_1^n = \Delta_1^{n-1}\Delta_2^{n-2}$$

Proof. Recall

$$\Delta_1^n = w_1^n \dots w_{F_{n-1}}^n$$
$$\Delta_1^{n-1}\Delta_2^{n-1} = w_1^{n-1} \dots w_{F_{n-1}}^{n-1}$$

Thus we must prove for all $n, k \in \mathbb{N}_0$ with $n > 2$ and $k < F_{n-1}$,

$$w_k^n = w_k^{n-1}.$$

We will proceed by induction on n. The case $n = 3$ can be verified directly from Figure 3. Fix n and suppose that for $k \in \mathbb{N}_0$ with $k < F_n$

$$w_k^n = w_k^{n-1}.$$

We consider two cases: $w_k^n = 1$ and $w_k^n = 0$. First we suppose that $w_k^n = w_k^{n-1} = 1$. Using 7 we get

$$0 < kF_{n-1} \pmod{F_n} \leq F_{n-2}$$

and

$$0 < kF_{n-2} \pmod{F_{n-1}} \leq F_{n-3}.$$

Summing and applying Lemma 8 we get:

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-1}.$$

Hence $w_k^n = 1$ by 7.

Now suppose $w_k^{n-1} = 0$. Thus

$$F_{n-2} < kF_{n-1} \pmod{F_n}$$

and

$$F_{n-3} < kF_{n-2} \pmod{F_{n-1}}$$

Summing and applying Lemma 8 we get

$$F_{n-1} < kF_n \pmod{F_{n+1}}$$

and so $w_k^n = 0$. In either case $w_k^n = w_k^{n-1}$ concluding the proof. □

4.4. **Proof of Lemma 13.**

Lemma 13. *Let $n \in \mathbb{N}$ with $n > 2$ then*

$$\Delta_3^n = \begin{cases} \overline{\Delta_1^{n-1}}\Delta_2^{n-1} & n \text{ odd} \\ \Delta_1^{n-1}\Delta_2^{n-1} & n \text{ even} \end{cases}$$

Proof. Fix $n \in \mathbb{N}$ with $n > 2$. By Lemma 6 this is the same as proving that for all k with $F_n < k \leq F_{n+1}$ with $k \neq F_n + 2$

$$w_k^n = w_{k-F_n}^n$$

while

$$w_{F_{n+2}}^n = \begin{cases} w_2^n & n \text{ odd} \\ \overline{w_2^n} & n \text{ even.} \end{cases}$$

Fix $k \in \mathbb{N}$ with $F_n < k < F_{n+1}$, (F_{n+1} is handled later). Set $p = k - F_n$ and note $p \leq F_{n-1}$. Using d'Ocagne's identity, compute

$$\begin{aligned}(p + F_n)F_n \pmod{F_{n+1}} &= (pF_n + F_n^2) \pmod{F_{n+1}} \\ &= (pF_n + (-1)^{n-1}) \pmod{F_{n+1}}.\end{aligned}$$

We now consider the two cases: n is even and n is odd. Suppose first that n is even. We wish to prove that $w_k^n = w_p^n$.

Suppose that $w_p^n = 1$ then we know that

$$0 < pF_n \pmod{F_{n+1}} \leq F_{n-2}$$

by (7). Adding $-1 \pmod{F_{n+1}}$ gives us

$$-1 \pmod{F_{n+1}} < kF_n \pmod{F_{n+1}} \leq F_{n-2} \tag{12}$$

In addition notice we assumed $k < F_{n+1}$ and so by Lemma 4 we know that $kF_n \pmod{F_{n+1}} \neq 0$ as $F_{n+1}F_n \pmod{F_{n+1}} = 0$. Thus we can rewrite 12 as

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2} \tag{13}$$

and conclude $w_k^n = 1 = w_p^n$ as desired. Now assume that $w_p^n = 0$.

$$F_{n-2} < pF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}.$$

Adding $-1 \pmod{F_{n+1}}$,

$$F_{n-2} \leq kF_n \pmod{F_{n+1}} \leq -2 \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}.$$

Again note that $k > F_n$ and so we know that $kF_n \pmod{F_{n+1}} \neq F_{n-2}$ as $2F_n \pmod{F_{n+1}} = F_{n-2}$. Thus

$$F_{n-2} < kF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}.$$

as $w_k^n = 0 = w_p^n$.

Now assume that n is odd. We repeat the same process. Suppose that $w_p^n = 1$. Then

$$0 < pF_n \pmod{F_{n+1}} \leq F_{n-2}$$

by (7). Adding $1 \pmod{F_{n+1}}$ gives us

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2} + 1. \tag{14}$$

Note that if $k = F_n + 2$ then $p = 2$ and so $pF_n \pmod{F_{n+1}} = F_{n-2}$ and $kF_n \pmod{F_{n+1}} = F_{n-2} + 1$. Hence if $k \neq F_n + 2$ then

$$0 < kF_n \pmod{F_{n+1}} \leq F_{n-2}$$

and $w_k^n = 1 = w_p^n$. If $k = F_n + 2$ then

$$kF_n \pmod{F_{n+1}} = F_{n-2} + 1 > F_{n-2}$$

and so $w_k^n = 0 = \overline{w_p^n}$ as desired.

Lastly suppose $w_p^n = 0$. Then

$$F_{n-2} < pF_n \pmod{F_{n+1}} \leq -1 \pmod{F_{n+1}}.$$

Adding $1 \pmod{F_{n+1}}$,

$$F_{n-2} < F_{n-2} + 1 < kF_n \pmod{F_{n+1}} \leq 0 \pmod{F_{n+1}}.$$

Again note that $kF_n \pmod{F_{n+1}} \neq 0$ and so we can conclude

$$F_{n-2} < kF_n \pmod{F_{n+1}} < 0 \pmod{F_{n+1}}.$$

and so $w_k^n = 0 = w_p^n$ as desired. Lastly we handle the special case when $k = F_{n+1}$. Recall that by definition $w_{F_{n+1}}^n = 1$ if n is even and 0 if n is odd. We wish to show that $w_{F_{n-1}}^n = w_{F_{n+1}}^n$. Notice

$$F_{n_1}F_n \pmod{F_{n+1}} = (-1)^n \pmod{F_{n+1}}$$

In particular, if n is even then $w_{F_{n-1}}^n = 1$ and if n is odd $w_{F_{n-1}}^n = 0$. Hence $w_{F_{n-1}}^n = w_{F_{n+1}}^n$. This concludes the proof. $\square$

4.5. **Lemma 14.**

Lemma 14. *Let $n \in \mathbb{N}$ with $n > 2$ then*

$$\Delta_2^n = \begin{cases} \overline{\Delta_1^{n-1}} & n \text{ even} \\ \Delta_1^{n-1} & n \text{ odd} \end{cases}$$

Proof. We use the same argument as the previous lemma. Fix $n \in \mathbb{N}$ with $n > 2$, we wish to show that for all $k \in \mathbb{N}$ with $F_{n-1} < k \leq F_n$

$$w_k^n = w_{k-F_{n-1}}^n.$$

Fix k such that $F_{n-1} < k \leq F_n$. Set $p = k - F_{n-1}$ and compute

$$\begin{aligned} kF_n \pmod{F_{n+1}} &= (p + F_{n-1})F_n \pmod{F_n + 1} \\ &= (pF_n + F_{n-1}F_n) \pmod{F_{n+1}} \\ &= (pF_n + (-1)^n) \pmod{F_{n+1}}. \end{aligned}$$

Where the last line follows from d'Ocagne's identity with $p = n$ and $q = n - 2$ as follows

$$\begin{aligned} F_nF_{n-1} - F_{n+1}F_{n-2} &= (-1)^n F_2 \\ F_nF_{n-1} \pmod{F_{n+1}} &= (-1)^n \pmod{F_{n+1}}. \end{aligned}$$

From here the argument proceeds the same as the previous lemma with the cases reversed. $\square$

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