MATH 403 (PROBABILITY), AUG 2023 PRELIM

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This is an open notes prelim.

Problem 1. Suppose $\{X_n\}_{n=1}^{\infty}$ are iid random variables such that $\mathbb{E}[|X_1|] = +\infty$. Show

$$\varlimsup_{n\to\infty} \frac{S_n}{n} = +\infty \tag{1}$$

where $S_n = \sum_{i=1}^{n} X_i$. *Hint: Consider the inequality* $|X_n| \le |S_n| + |S_{n-1}|$ *and first see what happens to* $\varlimsup |X_n|/n$. *Is* $|X_n|/n$ *large fairly regularly?*

Problem 2. Suppose $\{X_n\}_{n=1}^{\infty}$ are iid Cauchy random variables with density

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad x \in \mathbb{R}$$

(1) Compute $\mathbb{E}[|X_1|]$, and find $\varlimsup_{n\to\infty} S_n/n$.
(2) Compute the characteristic function $\phi(t)$ of X_1. *Hint: Consider using the residue theorem or computing the inverse Fourier transform of* $e^{-|t|}$.
(3) Does S_n/n have a weak limit?

Problem 3. Construct a sequence such that $X_n \to X$ in distribution but $X_n \not\to X$ in measure.

Suppose $F_n(t) \to F(t)$ for all $t \neq c$, where F_n is the cumulative distribution function of X_n and F is the cdf given by

$$F(t) = \begin{cases} 1 & t \ge c \\ 0 & t < c \end{cases}$$

where $c \in \mathbb{R}$. Show that $X_n \to c$ in measure.

Problem 4. Let $Y_1, Y_2, \ldots$ be nonconstant, nonnegative, iid random variables with $\mathbb{E}Y_m = 1$.

(1) Show that
$$X_n = \prod_{m\le n} Y_m$$
defines a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$.
(2) The martingale convergence theorem tells us that there is an X_∞ such that $X_n \to X_\infty$ $\mathbb{P}$ a.s. Determine X_∞. *Hint: Consider using the law of large numbers.*

Problem 5. Let p be a fixed number in $[1, \infty]$. Let X_n be a sequence of random variables such that for every $\epsilon > 0$, there exists an N such that for all $n, m \ge N$, $\mathbb{E}[|X_n - X_m|^p] < \epsilon$. Show that there is an X such that $X_n \to X$ in probability.