COMPLEX ANALYSIS PRELIMINARY EXAMINATION

Please write your solutions in the blue books that are provided. If you choose to use a major theorem, state it and label it clearly, but do not prove it unless otherwise indicated.

Problem #1: Let f be a complex valued function on the unit disk such that both $g = f^2$ and $h = f^3$ are analytic. Prove that f is analytic on the unit disk.

Problem #2: For each positive integer $n \geq 2$ compute

$$\int_0^\infty \frac{dx}{1+x^n}.$$

Hint: Consider a contour consisting of a piece of the positive real axis, a piece of the line pointing in the direction of the n'th primitive root of unity, and the connecting circular arc.

Problem #3: Show that if f is analytic on an open set containing the closure of the disk of radius R centered at a, then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \left|f(a + re^{i\theta})\right|^2 r\,dr\,d\theta.$$

Problem #4: Prove that for any positive integer n and any $a \in \mathbb{C}$, the polynomial $az^n + z + 1$ has at least one root in the disk $|z| \leq 2$.

Hint: Write $az^n + z + 1 = a(z - \omega_1) \cdot \cdots \cdot (z - \omega_n)$, with justification, and compare the constant terms.

Problem #5: Let $f : D \to D$ be a holomorphic function on the unit disk. Prove that for all $z \in D$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Hint: You may want to use the fact that the transformation

$$z \to \frac{w - z}{1 - \overline{w}z}$$

maps D to D and swaps w and the origin.

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