Real Analysis Preliminary Exam, August 2021

1. Let $(X, \mathcal{S})$ be a measurable space and assume $(f_n)_{n\geq 1}$ is a sequence of real-valued, $\mathcal{S}$-measurable functions. Prove that

$$\{x \in X; (f_n(x))_{n\geq 1} \text{ is convergent}\} \in \mathcal{S}.$$

2. Prove that if $(X, \mathcal{S}, \mu)$ is a measure space and $f \in \mathcal{L}^1(\mu)$ then

$$\lim_{n\to\infty} n \cdot \mu(\{x \in X; |f(x)| \geq n\}) = 0.$$

3. If $h : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function, then its associated Hardy-Littlewood maximal function $h^* : \mathbb{R} \to [0, \infty]$ is defined by

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{[b-t,b+t]} |h(x)|\, d\lambda(x), \qquad (\forall)\, b \in \mathbb{R}.$$

Prove that if $g : \mathbb{R} \to [0, \infty)$ is increasing, then g^* is increasing. (Note: $g : \mathbb{R} \to \mathbb{R}$ is an increasing function if $a < b$ implies $g(a) \leq g(b)$.)

4. Prove using the results concerning iterated integrals (i.e., you should write the integral below as a a double integral) that

$$\int_0^\infty e^{-x} \frac{\sin^2 x}{x}\, dx = \frac{\ln 5}{4}.$$

(Note: In arguing for this problem, you may assume without proof that a continuous function $h : X \times Y \to \mathbb{R}$ is measurable with respect to the product measure if X and Y are Borel subsets of $\mathbb{R}$.)

5. Let $E = (0, \infty)$ and consider $f : E \to \mathbb{R}$ given by

$$f(x) = \frac{x^{-\frac{1}{2}}}{1 + |\ln x|}, \qquad (\forall)\, x \in E.$$

Show that $f \in L^p(E)$, where $1 \leq p \leq \infty$, if and only if $p = 2$.