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1a) Define the n^{th} cyclotomic polynomial $\Phi_n(x)$ over an arbitrary field k where the characteristic of k is either 0 or a prime p not dividing n.

b) Prove that $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

c) Give an example if possible, and briefly explain why your example works. If no such example exists, briefly explain why this is so.

i) an integer $n \geq 5$ satisfying the property that $\Phi_n(x)$ is ***irreducible*** over $\mathbb{F}_p$ for all primes p not dividing n.

ii) an integer $n \geq 5$ satisfying the property that $\Phi_n(x)$ is ***reducible*** over $\mathbb{F}_p$ for all primes p not dividing n.

2) This relates to material in the book, not to a HW problem. To answer this question, you can use the additive form of Hilbert's Theorem 90 without proof.

Let k be a field in characteristic $p \neq 0$. Prove each of the following.

a) Let K be a cyclic extension of k of degree p. Then $K = k(\alpha)$ for some $\alpha \in K$ that is a root of a polynomial $x^p - x - a$ for some $a \in k$.

b) Conversely, for any $a \in k$, the polynomial $x^p - x - a$ either has one root in k, in which case, all its roots are in k, or it is irreducible. Moreover, in the latter case, $k(\alpha)$ is Galois and cyclic of degree p over k.

3,4,5) For purposes of the prelims, your best 3 of the following 4 problems *(A, B, C, or D, some of which have multiple parts, all of which were homework questions or are very related to ones that were)* will count. For purposes of the class final, all 4 will count, but you do not have to complete all 4 problems, as the scores will be curved.

Ai) Let L be an extension of a field k, and let $\alpha, \beta \in L$ be algebraic over k. Suppose that $k(\alpha)$ is a separable extension of k and that $k(\beta)$ is a purely inseparable extension of k.

a) Show that if F is an intermediate field between L and k, then $F(\alpha)$ is a separable extension of F.

b) Show that if E is an intermediate field between L and k, then $E(\beta)$ is a purely inseparable extension of E.

c) Show that $k(\alpha + \beta) = k(\alpha, \beta)$

ii) Give an example of a purely inseparable extension of fields of degree bigger than 1.

Bi) Suppose E and L are finite extensions of a field k contained in the same algebraic closure of k. For each of the following, provide a brief proof if the statement is true, and a counterexample if the statement is false.

a) $[EL\colon E]$ must divide $[L\colon k]$

b) If $[EL\colon E] = [L\colon k]$, then $E \cap L = k$

c) If $E \cap L = k$, then $[EL\colon E] = [L\colon k]$.

ii) Would your answers to part i change any under the added hypothesis that L is Galois over k? Briefly explain why this is so.

C) Suppose that E is an ***algebraic*** extension of k for which every non-constant polynomial in $k[x]$ has at least one root in E. Prove that E is algebraically closed.

Di) Let K be a field with no nontrivial abelian Galois extensions. Suppose that n is a positive integer and either $\mathrm{char}(K) = 0$ or n is relatively prime to $\mathrm{char}(K)$. Prove that every element of K is an n^{th} power in K.

ii) Is the statement still true if n is not relatively prime to $\mathrm{char}(K)$? Prove it or give a counterexample.