Math 436 Final Exam/Algebra 1 Prelims-December 2021

1i) For each of the ***numerical*** values $d_1 = 5 \cdot 7 \cdot 29$ and $d_2 = 59 \cdot 11 \cdot 3$, tell whether or not a group of order d_i must be cyclic. If the answer is yes, prove it *(without appealing to any general theorems about groups of order pqr for arbitrary primes p, q, r)*, and if the answer is no, give a counterexample, or explain how you know one must exist.

ii) Prove that any group of order p^2q *($p \neq q$ prime)* is solvable and one of its Sylow groups must be normal. *(Make sure your argument also works in the case p=2 and q=3.)*

2,3) Answer two of the following three problems (***A, B, C***). Each problem has multiple parts *(for purposes of the final exam, I will count all, parts but for purposes of the prelims I will only count 2-if you attempt to do them all, I will count the ones you do best).*

Ai) Suppose a finite p-group acts on a finite set $\mathcal{S}$. Prove that the number of fixed points is congruent to the cardinality of $\mathcal{S}$ mod p.

ii)Use part i to prove that if G is a finite p group, the center of G cannot be trivial. Be sure to explain how you are using part i.

iii) Again suppose G is a finite p-group. What can you deduce from Part i about the number of subgroups of G?

B) Let D_{2n} be the dihedral group of order $2n$, with generators σ and τ of orders n and 2 respectively.

i) 8 pts. Show that σ^2 belongs to the commutator subgroup of D_{2n}.

ii) 12 pts. If n is odd *(resp. even)* show that the commutator subgroup of D_{2n} equals $\langle\sigma\rangle$(resp. $\langle\sigma^2\rangle$). *(You **must** completely justify all of your conclusions, and also must give a solution not involving **any** computations for this part, except you can use the result of Part a, and also use the fact that $\langle\sigma^2\rangle \lhd G$ without proof.).*

Ci) Show that $\mathbb{Q}/\mathbb{Z}$ is a torsion group that has exactly 1 subgroup of each order $n \geq 1$, and that each such subgroup must be cyclic.

ii) Deduce from this result that $\mathbb{Q}/\mathbb{Z}$ can **not** be written as the direct sum or direct product of cyclic groups *(either of a finite number of such groups, or of an infinite number).*

iii) Let A be the commutative ring formed by all infinite sequences of rational numbers that are constant after a point, i.e. all $(a_1, a_2, a_3, \ldots)$ such that $a_n = a_{n+1} = a_{n+2} = \cdots$ from some point on. Find all maximal ideals in A and justify your conclusion.

4i) Give an example of a commutative ring A and a multiplicatively closed subset S satisfying $0 \notin S$ such that all prime ideals of A intersect S, if some such example exists. If there are no examples, briefly explain why not.

ii) Let A be an integral domain. Show that $\bigcap_m A_m = A$ (where the A_m are all viewed as being subrings of the quotient field K of A and the intersection is taken over all maximal ideals m in A). *Warning. You can **not** simply argue that if $\frac{a}{b} \in A_m$, then $b \notin m$, as this is false.*

5) Let A be a Noetherian ring (commutative with $1 \neq 0$).

i) Use the maximality property of Noetherian rings to show that every ideal of A must contain a finite product of *(not necessarily distinct)* prime ideals.

ii) Deduce that in any Noetherian ring A, (0) equals a finite product of prime ideals.

iii) Now assume that all prime ideals are maximal in A. Use part b to show that A has only a finite number of prime ideals. *(Do not use the theory of Artin rings in your answer.)*