ALGEBRA I PRELIM, DECEMBER 2013
(SEE THE NEXT PAGE)

(Analysis and Topology prelims follow)

1a) If $f : G \longrightarrow G'$ is a group homomorphism and H is a subgroup of G, is it always the case that $f^{-1}(f(H)) = H$ (where $f^{-1}(f(H))$ denotes the inverse image of $f(H)$ in G or equivalently the set of all elements in G that map into $f(H)$)? Prove it or give a counterexample.

b) If the answer to part a is no, then give a complete description of the subgroup of G that equals $f^{-1}(f(H))$ and apply this description to your counterexample. If the answer to part a is yes, then you can skip this part of the question.

2) Let G^c denote the commutator subgroup of a group G.

a) Let N be any normal subgroup of G. Prove that G/N is abelian if and only if $G^c \subseteq N$.

b) Show if H is any subgroup of G such that $G^c \subseteq H$, then H is a normal subgroup of G.

c) Recall that a group G is called solvable if there exists a normal tower of subgroups starting with G and ending with $\{e\}$ for which all the factor groups are abelian. Now define the following subgroups of G, $G^{(0)} = G$, $G^{(1)} = G^c$ =commutator subgroup of G, $G^{(2)} = (G^{(1)})^c$ =commutator subgroup of $G^{(1)}$, $G^{(3)} = (G^{(2)})^c$, etc. Show that G is solvable if and only if $G^{(s)} = \{e\}$ for some positive integrer s.

3a) Suppose that $K \lhd H \lhd G$ and K is a Sylow p-subgroup of H for some prime p, prove that $K \lhd G$.

b) Suppose G is a finite group, P is a Sylow p-subgroup, and H a (not necessarily Sylow) p-subgroup of G. If $H \subseteq N_G(P)$, prove that $H \subseteq P$.

4*)* Let A be an integral domain and M an A module. An element $x \in M$ is called a torsion element of M if the annihilator of x, $\mathrm{Ann}(x)$, does not equal $\{0\}$.

(a) Show that the subset $T(M)$ of torsion elements of M forms a submodule of M. Would this still be the case, if A were not an integral domain? Prove it or give a counterexample.

(b) If $f : M \to N$ is a homomorphism of A-modules, show that $f(T(M)) \subseteq T(N)$.

(c) If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A-modules, show that $0 \to T(M') \xrightarrow{f'} T(M) \xrightarrow{g'} T(M'')$ is also exact where f' and g' are the restrictions of f and g to $T(M')$ and $T(M)$, respectively.

(d) Would the result in (c) still be true if we added a 0 to the right-hand side of the above two sequences? Prove it or give a counterexample.

5) For each of the following, give an example if possible, and if not possible, ***briefly*** explain why not. All rings in this problem are assumed to be commutative and all of your examples should involve only commutative rings. *For purposes of the prelims, you must get at least 5 of these correct in order to have the problem count as being correct.*

(a) A commutative ring A and a multiplicatively closed subset S which does not contain 0, such that$S^{-1}(A)$ is not a local ring.

(b) A commutative ring A and a multiplicatively closed subset S which does not contain 0, such that all prime ideals of A intersect S,

(c) A subring$A$ of a Noetherian ring B that is not a Noetherian ring.

(d) A ring A and a submodule of a finitely generated $A-$module that is not finitely generated.

(e) A non-Noetherian ring A that is a Noetherian $\mathbb{Z}$-module.

(f) A Noetherian ring A that is a non Noetherian $\mathbb{Z}$-module.

Analysis Preliminary Exam, Fall 2013

Problem #1: Let $f : [0,1] \to \mathbb{R}$.

i) Define what it means for f to be Lipschitz. Also define what it means for f to be **absolutely** continuous.

ii) Prove that if f is Lipschitz, then it is absolutely continuous.

iii) Prove that $f(x) = \sqrt{x}$ is absolutely continuous, but not Lipschitz on $[0,1]$.

Problem #2: Define what it means for a function $f : [a,b] \to \mathbb{R}$ to be of bounded variation. Prove that a function of bounded variation is a difference of two monotone functions.

Problem #3: Recall that $E \subset \mathbb{R}^d$ is Lebesgue measurable if for every $\epsilon > 0$ there exists an open set O containing E such that $m^*(O \backslash E) \leq \epsilon$, where m^* is the outer measure. Prove that $E \subset \mathbb{R}^d$ is Lebesgue measurable if and only if for every $A \subset \mathbb{R}^d$,

$$m^*(A) = m^*(A \cap E) + m^*(A \backslash E).$$

Problem #4: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $[a,b]$ and twice continuously differentiable on (a,b).

i) Suppose that $f''(x) > 0$ on (a,b). Prove that

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b).$$

ii) Use i) to show that if $x, y \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

iii) Use ii) to show that if f, g are absolutely integrable on $\mathbb{R}$, then

$$\int f(x)g(x)dx \leq \left(\int |f(x)|^p dx \right)^{\frac{1}{p}} \cdot \left(\int |g(x)|^q dx \right)^{\frac{1}{q}}.$$

Problem #5: Let $f : \mathbb{R}^d \to \mathbf{C}$ be an absolutely integrable function. Prove that

$$m\left(\left\{x \in \mathbb{R}^d : \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy > \lambda\right\}\right) \leq \frac{C}{\lambda}||f||_{L^1(\mathbb{R}^d)},$$

where m is the Lebesgue measure on $\mathbb{R}^d$ and the supremum is taken over all the dyadic cubes Q. Recall that a dyadic cube is a cube of the form

$$\left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \cdots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right]$$

for some integers $n, i_1, \ldots, i_d$.

Hint: Follow the proof of the Hardy-Littlewood maximal inequality. You may find special properties of dyadic cubes helpful in establishing an appropriate covering lemma.

MTH 440 – GENERAL TOPOLOGY — December 2013

I. Let A be a set. Let $\{X_\alpha\}_{\alpha\epsilon J}$ be a indexed family of spaces, and let $\{f_\alpha\}_{\alpha\epsilon J}$ be an indexed family of functions $f : A \to X_\alpha$.

(a) Show that in a unique smallest (coarsest) topology $\mathcal{T}$ on A such that each f_α is continuous.

(b) Let $S_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\}$ and $S = \cup_\beta S_\beta$. Show that S is a subbasis for the topology $\mathcal{T}$ on A.

(c) Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each composition $f_\alpha \circ g$ is continuous.

(d) Let $f : A \to \prod_{\alpha\epsilon J} X_\alpha$ be defined by

$$f(a) = (f_\alpha(a))_{\alpha\epsilon J}$$

and Z be the subspace $f(A)$ of the product space $\prod_\alpha X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open subset of Z.

II. Let X be a connected topological space. Let $x \in X$ be such that $X - \{x\}$ is disconnected with separation $X = A \cup B$. Prove $A \cup \{x\}$ is connected.

III. A topological space X is *countably compact* if every countable open covering of X contains a finite subcollection that covers X. Assume X is a Hausdorff space. Prove that the countably compact condition is equivalent to the (Bolzano-Weierstrass) condition that every infinite set in X has a limit point.

IV. Recall that a Baire space is characterized by the condition that any countable union of closed sets, each with empty interior, has empty interior. Prove that an open subspace of a Baire space is a Baire space.

V. Let $\{A_\alpha\}$ be a locally finite collection of subsets of a topological space X. This condition means that each point of X lies in an open set which has a non-empty intersection with at most finitely many of the A_α. Prove

$$\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A}_\alpha \ .$$