1 Algebra II

1. Prove that a Dedekind domain R is a UFD if and only if it is a PID.

2. Prove that $x^4 - 2$ is solvable in two ways: first by showing the splitting field is a radical field extension of $\mathbb{Q}$, and next by showing its Galois group is solvable.

3. Let R be a local ring and $h : R \to S$ a ring homomorphism. Prove that the image $h(R)$ is a local ring.

4. Recall *Noether's normalization theorem*: if B is a finitely generated k-algebra, where k is a field, then there exists a subset $\{y_1, \ldots, y_r\}$ of B such that the y_i are algebraically independent over k and B is integral over $k[y_1, \ldots, y_r]$.

   Prove *Zariski's lemma*: Let A be a finitely generated k-algebra. If I is a maximal ideal of A, then A/I is a finite extension of k. (One approach is to use Noether normalization on A/I.)

5. Let G be an abelian group. Prove that any irreducible representation of G is of order 1.

2 Complex Analysis

1. Let $f, g : D \to \mathbb{C}$ be two holomorphic functions defined on a domain $D \subset \mathbb{C}$ such that

$$f(z) + \overline{g(z)} \in \mathbb{R}, \qquad (\forall) z \in D.$$

Show that there exists necessarily a constant $A \in \mathbb{R}$, with

$$f(z) - g(z) = A, \qquad (\forall) z \in D.$$

2. Determine, with proof, whether there exist functions f which are holomorphic in a neighborhood of 0 and satisfy

$$n^{-5/2} < \left| f\left(\frac{1}{n}\right) \right| < 2n^{-5/2}, \qquad (\forall) n \geq 1.$$

3. For $a > 0$ fixed, compute, using the residue theorem and explaining all steps,

$$\int_0^\infty \frac{x^2}{(x^2 + a^2)^3}\, dx.$$

4. Prove that for all $\lambda > 1$ the equation

$$z = \lambda - e^{-z}$$

has precisely one root in the half-plane $\operatorname{Re} z \geq 0$.

6. Find a conformal transformation $w = f(z)$ which maps the angle $|\arg z| < \pi/4$ into the unit disk $|w| < 1$ and verifies

$$f(1) = 0, \quad \arg f'(1) = \pi.$$

3 Geometry

1. a) Explain why you need at least two coordinate charts to cover a compact manifold.

b) Check whether the following maps are local diffeomorphisms. Are they also global diffeomorphisms ? Explain why.

- $F(x, y) = (\sin(x^2 + y^2), \cos(x^2 + y^2))$
- $F(x, y) = (e^x \sin y, e^x \cos y)$
- $F(x, y) = (5x, ye^x)$

2. Let (r, θ) be the polar coordinates defined on $\mathbb{R}^2$ outside of the origin.

a) Write the 1-forms dr, $d\theta$ in terms of the ordinary coordinates x, y.

b) Write the volume form $dx \wedge dy$ in terms of dr and $d\theta$. (hint: use the pullback of differential forms.)

3. Show that the Laplacian Δ has the following properties :

a) $*\Delta = \Delta *$. (Show also that this property implies that if ω is a harmonic form, so is $*\omega$).

b) Δ is self-adjoint, that is $\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle$.

c) A necessary and sufficient condition for $\Delta\omega = 0$ is that $d\omega = 0$ and $d^*\omega = 0$.

d) Let M be connected, oriented, compact Riemannian manifold. Then a harmonic function on M is a constant function. Also if n= dim M, then a harmonic n-form is a constant multiple of the volume element $dvol_M$. (hint: use part (c))

4. (Proof of Hodge Theorem) Show that an arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form. In other words, show that the natural map $\mathbb{H}^k \to H^k_{DR}(M)$ is an isomorphism. Here $\mathbb{H}^k$ denotes the set of all harmonic k-forms on M and $H^k_{DR}(M)$ denotes the k-dimensional de Rham cohomology group of M.

(hint: use Hodge decomposition theorem which says that on an oriented compact Riemannian manifold, an arbitrary k form can be uniquely written as the sum of a harmonic form, an exact form and a dual exact form.)

5. (20pts) It is well known that the 4-sphere $\mathbb{S}^4$ has trivial k^{th} de Rham cohomology except for $k = 0$ and $k = 4$. Let α be a closed two-form on $\mathbb{S}^4$. Prove that $\alpha \wedge \alpha$ vanishes at some point.

6. Consider the two form $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$ and the vector field $v = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Show that the Lie Derivative of ω in the direction of v is zero. This means that ω is invariant under the flow of ϕ_t, the one parameter group of transformations generated by v.