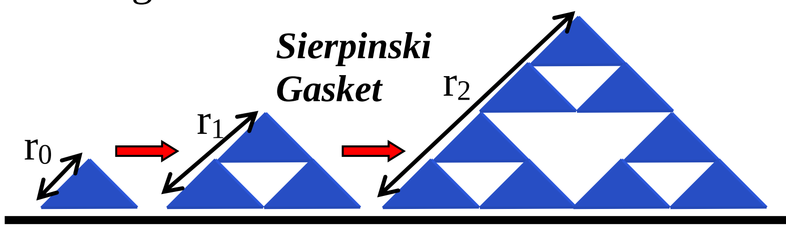
Fractal Structures and Dimension

Ordinary objects have a countable (integer) number of dimensions, $D = 0$ for a mathematical point, $D = 1$ for a line, $D = 2$ for a (hyper-) plane, and $D = 3$ for a volume in three-dimensional space. Fractal objects are called fractal, because their dimensionality is non-integer.

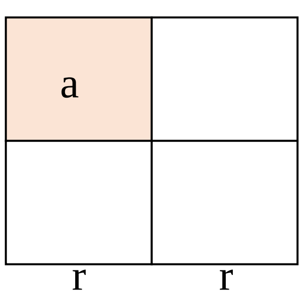


As an example, consider the Sierpinski Gasket illustrated above. Each iteration leads to a more complex entity which is built out of the previous structure, triangles stacked in a symmetric manner, producing another larger triangle with a more complex structure. The new triangle is scaled in size by a scaling factor $s = 2$ and it contains $N = 3$ pieces, each equal to the structure at the previous iteration. The law

$$N = s^D \tag{1}$$

connecting the number of pieces s with the dimensionality D is self-evident. The area of the new structure is obviously given by

$$A = N \cdot a \tag{2}$$



when a is the area of each piece.

These relations are intuitively clear for ordinary objects. For example, consider a normal plane object, a square shown in the figure on the left. If the side length of the square is scaled by a factor of $s = 2$, one obtains a 4-times larger area, which can be subdivided into $N = 4$ of the original pieces. Since $N = 2^2$, Equ. 1 applies with a scaling factor of $s = 2$ and the dimensionality of $D = 2$, as can be expected for any two-dimensional object.

For the Sierpinski Gasket, the number of pieces and scaling factor relate as

$$N = 3 = s^D = 2^D \tag{3}$$

which is equivalent to

$$\log 3 = D \cdot \log 2 \tag{4}$$

In other words, the Sierpinski Gasket has the fractal dimension

$$D = \log 3 / \log 2 = 1.5850 \tag{5}$$

The area of the structure is given by Equ. 2:

$$A = N \cdot a = s^D \cdot a \tag{6}$$

as a power law. Now, one can write this growth law as a general iteration n:

$$A_n = N \cdot A_{n-1} = s^D \cdot A_{n-1} = s^D \cdot \left(s^D \cdot A_{n-1}\right) = ... = s^{n \cdot D} \cdot A_0 \tag{7}$$

Taking the logarithms, this is equivalent to

$$\log A_n = \log s^{n \cdot D} + \log A_0 = D \cdot \log s^n + \log A_0 \tag{8}$$

Here, $A_0 = (1/2) r_0^2 = a$ is the area of the original triangle at the starting point $n = 0$ of the iteration. Remembering that the definition of the scale factor $s = r_n/r_{n-1} = r_{n-1}/r_{n-2} = \cdots = (r_1/r_0)^{\mathbf{n}}$, one can transform Equ. 8 to

$$\log A_n = \log s^{n \cdot D} + \log A_0 = D \cdot \log\left(\frac{r_n}{r_0}\right) + \log A_0$$
$$= D \cdot \log r_n - D \cdot \log r_0 - \log 2 + 2 \cdot \log r_0 \tag{9}$$

The density of the Sierpinski Gasket is defined as the filled area divided by the outlined area:

$$\rho_n = A_n \Big/ \left(\frac{1}{2} r_n^2\right) \tag{10}$$

Taking the log of this expression and inserting Equ. 9, one obtains

$$\log \rho_n = \log A_n - \log\left(\frac{1}{2} r_n^2\right) =$$
$$= D \cdot \log r_n - D \cdot \log r_0 - \log 2 + 2 \cdot \log r_0 - \log\left(\frac{1}{2} r_n^2\right)$$
$$= D \cdot \log r_n - D \cdot \log r_0 + 2 \cdot \log r_0 - 2 \cdot \log r_n =$$
$$= (2 - D) \cdot \log r_0 - (2 - D) \cdot \log r_n \tag{11}$$

Hence, one expects a linear relation between the logarithmic density ρ and linear dimension r of the fractal, here of the Sierpinski Gasket of the form

$$\log \rho = \alpha - \beta \cdot \log r \tag{12}$$

with two positive constants α and β depending on the fractal dimension. Such laws can be derived for other fractal objects as well: The logarithm of the density ρ is a logarithmic straight line when plotted as a function of the characteristic dimension *r*. Mathematically, this is a so-called ***power law***.