Agenda: Complex Processes in Nature and Laboratory

Systems and dynamics, qualifiers
- Examples (climate, planetary motion),

Order and Chaos, determinism and stochastic unpredictability
- 1D dynamics: phase space curves/orbits

Non-linear dynamics in nature and their modeling
- Mathematical model (logistic map, climate,.....)
- Stability criteria, stationary states

Self replicating structures out of simplicity
- Cellular automata and fractal structures,
- Self-organization in coupled chemical reactions

Thermodynamic states and their transformations
- Collective and chaotic multi-dimensional systems
- Energy types equilibration,
- flow of heat and radiation

Reading Assignments
Weeks 1&2

LN I: Complex processes

Kondepudi Ch.19
Additional Material
J.L. Schiff:
Cellular Automata,
Ch.1, Ch. 3.1-3.6

McQuarrie & Simon
Math Chapters
MC B, C, D,

Intermediate Summary

Linear force laws are deterministic → lead to predictable time evolution and are not sensitive to initial conditions: Small changes in initial conditions → small changes in final positions and momenta. 1D appr. $f\left(x+\Delta x\right)\approx f\left(x\right)+\Delta x\cdot f'\left(x\right):=g\left(x\right)$

$$\textit{Stepwise iteration } f_{i+1}=g_i \textit{ with In.Cond.: } i=0,\ g_0=g\left(x_0\right)$$

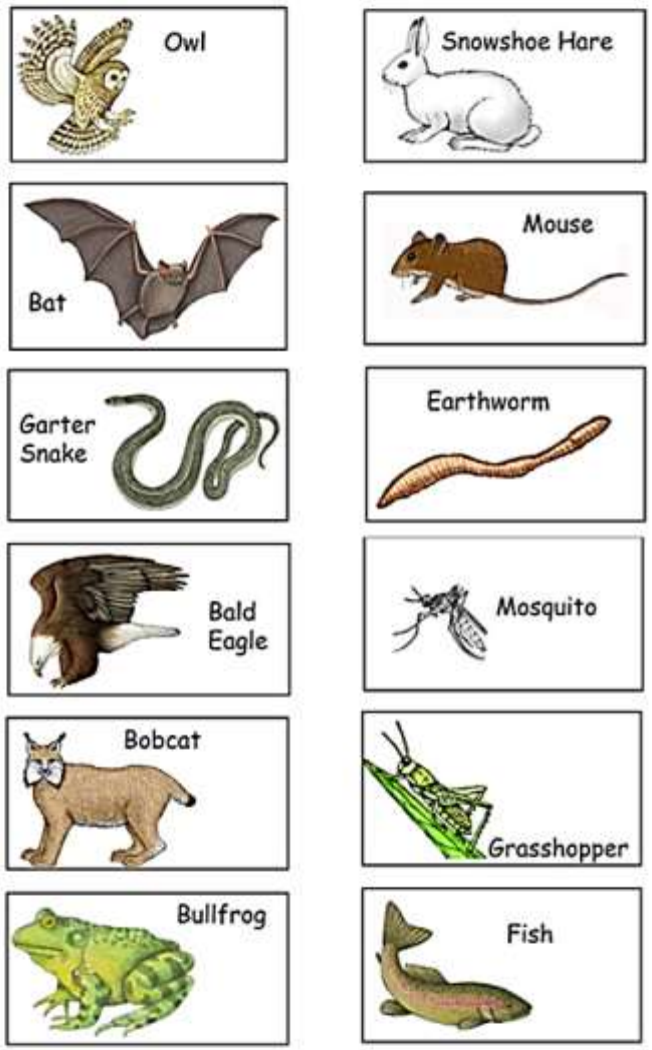
Trajectory $x\left(t\right)\to x_i:=x(t_i);\ \ \dot{x}\left(t\right)\to \dot{x}_i:=\dot{x}(t_i)$ *Phase curve* $\left\{x(t_i),\dot{x}(t_i)\right\};\ i=0,...$

- Complex, unpredictable (Chaotic) dynamics can be caused by non-linear force laws *for certain complex systems*, which have specific sets of properties (→model parameters).
- Complex, unpredictable (Chaotic) dynamics can be caused by correlated motion along different degrees of freedom. Rate equations for different d.o.f. become substantially entangled.
- Unpredictable ("chaotic") dynamics has high sensitivity to initial conditions.
- Chaotic dynamics can produce couplings (transitions) between different classes of periodic or (quasi-) random asymptotic states (vibrations/oscillations ←→ rotations).

Important examples: global climate, 3-body dynamics, forced mechanical or molecular oscillators, biological population dynamics, organ functionality, catalytic chemical reactions.

Analyze a simple (1D) chaotic system (climate rad balance, electric circuits)

Non-Linear Predator-Prey Dynamics



Predator-Prey competition for survival

Populations:

Species A $\rightarrow x(t) \leq 1$; *reproduction rate a*

Species B $\rightarrow y(t) \leq 1$; *reproduction rate b*

$$\dot{x} = \frac{dx}{dt} = (a-b)\,x \cdot [1-x]$$

$$\dot{y} = \frac{dy}{dt} = (b-a)\,y \cdot [1-y]$$

For small populations

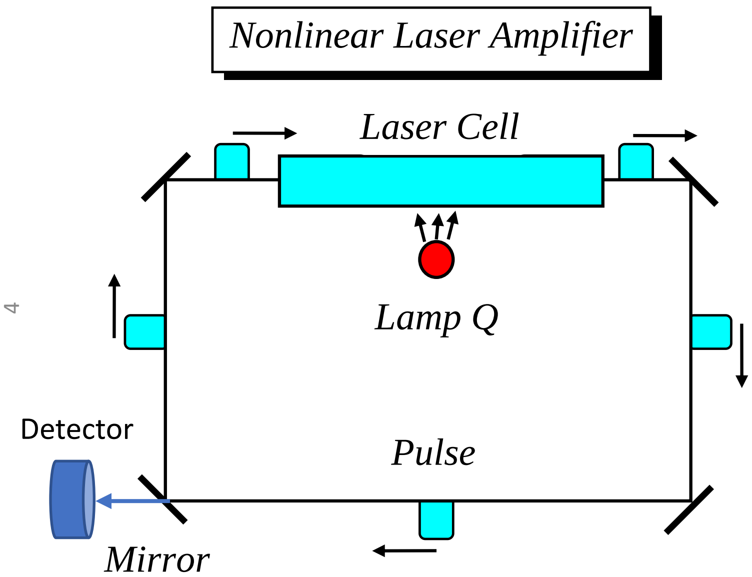
$$x \ll x_{max} := 1 \rightarrow x(t) \propto exp\{-(a-b)\cdot t\}$$

$$y \ll y_{max} := 1 \rightarrow y(t) \propto exp\{-(b-a)\cdot t\}$$

$$\dot{x} = a \cdot x^2 - \phi(x,y) \cdot x$$

$$\dot{y} = b \cdot y^2 - \phi(x,y) \cdot y$$

Laboratory Experiments On Complex (Chaotic) Dynamics



To investigate expected behavior of physical system → study mathematical properties of profile function and associated maps.
→ Test with laboratory experiments.

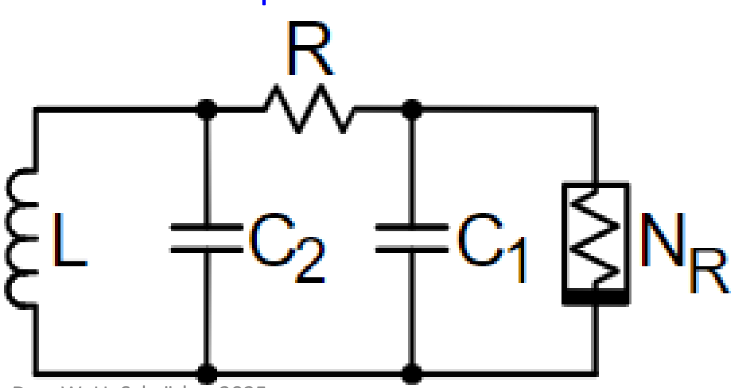
Initial maximum laser cavity intensity $I = 1$
Once around the track → $I_0 < 1$ → cavity
Stimulated emission ∝ **product of**
trigger intensity **x** available inversion

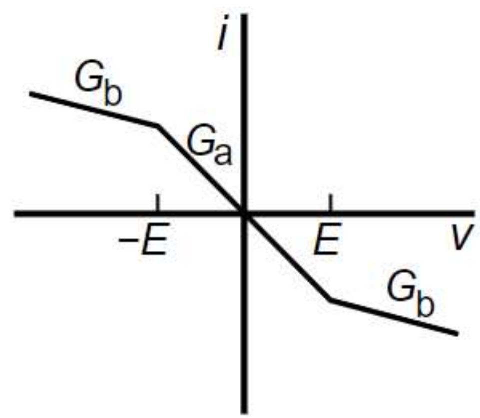
$$I_1 = \mu \cdot I_0 \cdot (1 - I_0) \quad etc \; n > 0$$

Logistic Map

n = number of circuits completed

Chua's Nonperiodic Oscillator

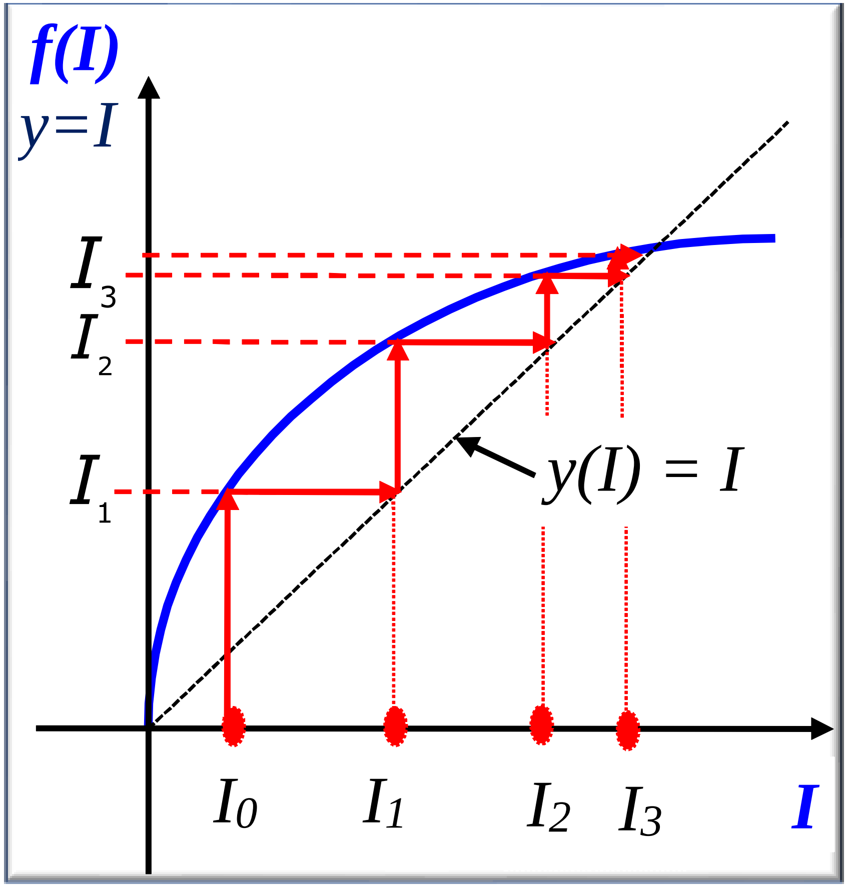




Chua Diode N_R : nonlinear negative resistance = amplifier with positive feedback.

Graphing An Iteration ("Cobweb Plot")

Sequence $I, f(I), f^2(I),\ldots,f^n(I)\ldots$
Plotted in 2D : $f(I_n)$ vs. I_n

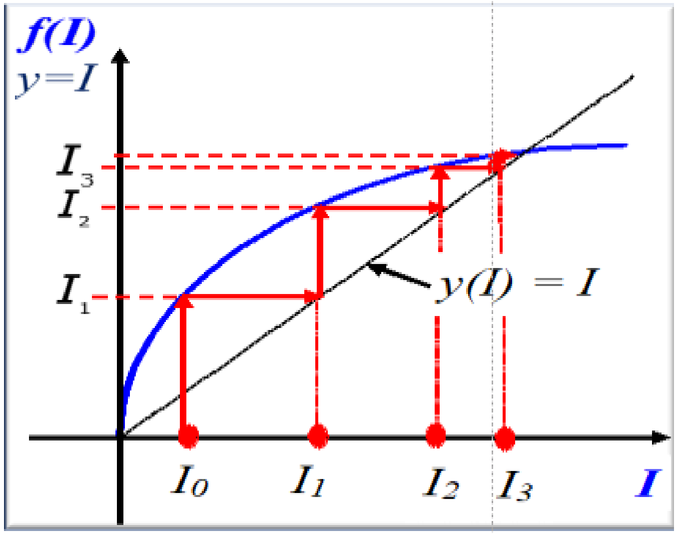


1. Draw horizontal (***I***) and vertical (***f***) axes of a 2D Cartesian coordinate system, with equal divisions.
2. Plot the map profile function *f(I) vs.* ***I .***
3. Plot the diagonal line ***y(I)*** = ***I.***

4. Start drawing the trajectory I_n , ($n = 0, 1\ldots\ldots$) by marking the initial point $I_{n=0}$ on the horizontal axis.
5. Draw a vertical arrow, from point I_n, to its functional value $I_{n+1} = f(I_n)$ on the profile curve.
6. Draw a horizontal arrow from point $f(I_n)$ to the point $f(I_n)= I_n$ on the $y = I$ line. This identifies the abscissa coordinate I_n for the next iteration.

7. Go to *5*) and repeat *5*) and *6*) until done.

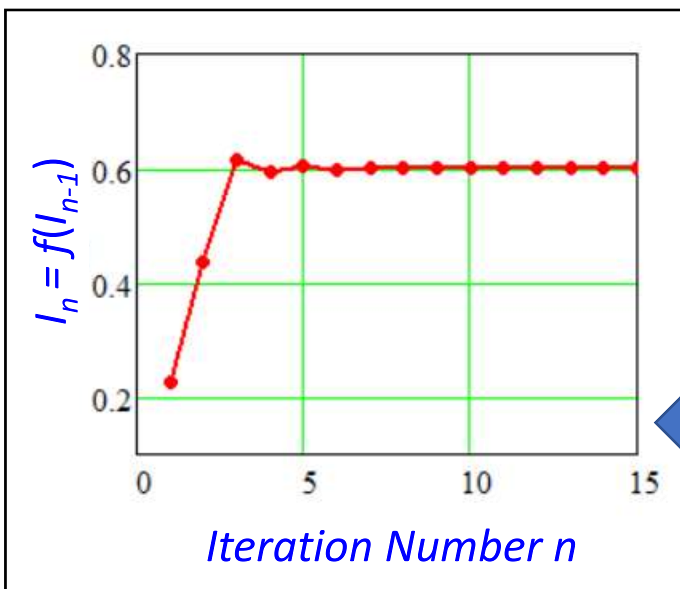
Graphing An Iteration II



Sequence I, $f(I)$, $f^2(I)$,…,$f^n(I)$…
Plotted in 2D

$f(I_n)$ vs. I_n

Different In : Laser intensity flickers

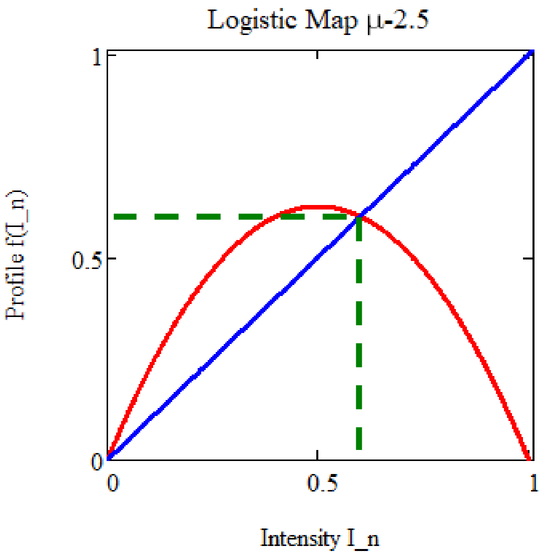


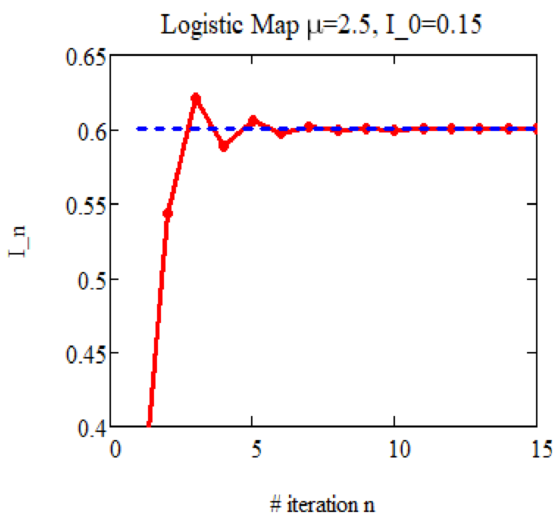
Sequence I, $f(I)$, $f^2(I)$,…,$f^n(I)$…
Plotted in 1D *vs.* I

Intensity I_n vs. Iteration number n

Intensity increases at first, then oscillates slightly. Finally, gets to steady-state operation after a few initial circuits (periods).

Logistic Map Features





Features of an iteration on a map depend on the profile function *f*, specifically on the amplification factor μ and the initial conditions,
InCon for 1D: just the starting point I_0.

$$\text{Periodic point } I_{pm}\text{, period } m: f^m(I_{pm}) = I_{pm}$$

$$\text{Fixpoints } I_f : f(I_f) = I_f \qquad \text{Trivial } I_f = 0$$

$$\text{Non-trivial FP exist if } f(I) \text{ and } y(I) = I \text{ intersect}$$

$$\text{Condition } \quad I_f = \frac{\mu - 1}{\mu} \geq 0$$

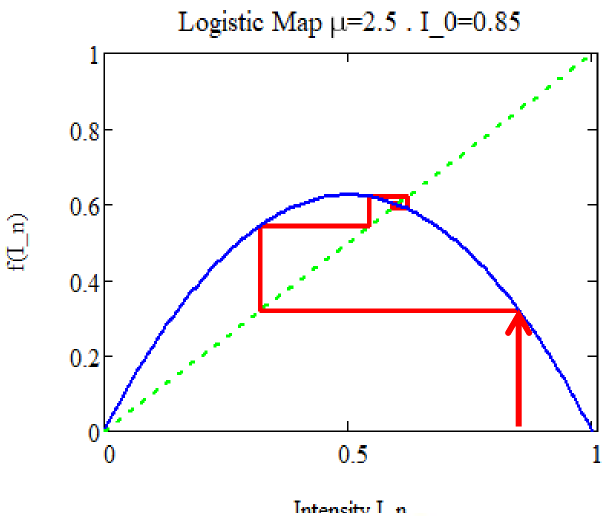
Trajectory ensembles with $I_0 \approx I_f$
fixpoints "attract" or "repel" (scatter)

$$\left| \left(\frac{df}{dI} \right)_{I_f} \right| < 1 \quad (I_f = \text{Attractor}) \qquad \left| \left(\frac{df}{dI} \right)_{I_f} \right| > 1 \quad (I_f = \text{Repellor})$$

$$\left| \left(\frac{df}{dI} \right)_{I_f} \right| = 0 \quad (I_f = ???)$$

Chaotic behavior if sensitivity to initial condition.

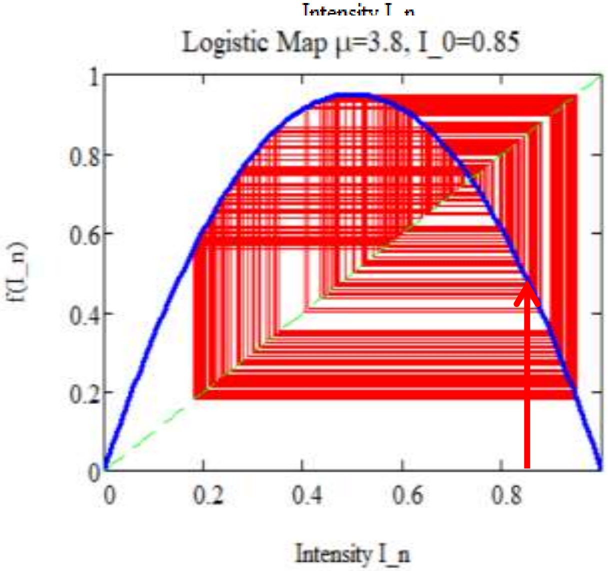
Order and Chaos Parameter Dependence

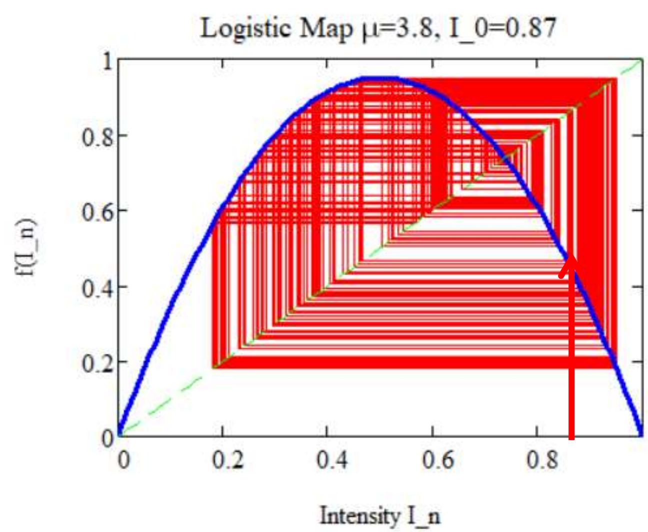


μ = 2.5: Fixpoint = attractor. All trajectories end up in this point: Laser operation stable after startup.

μ = 3.8 Fixpoint = strange attractor.
Trajectories spiral initially around fixpoint: intensity blinks slightly. After a few cycles, oscillations between 3 and 4 different brightness levels, highly unstable, essentially right after start.

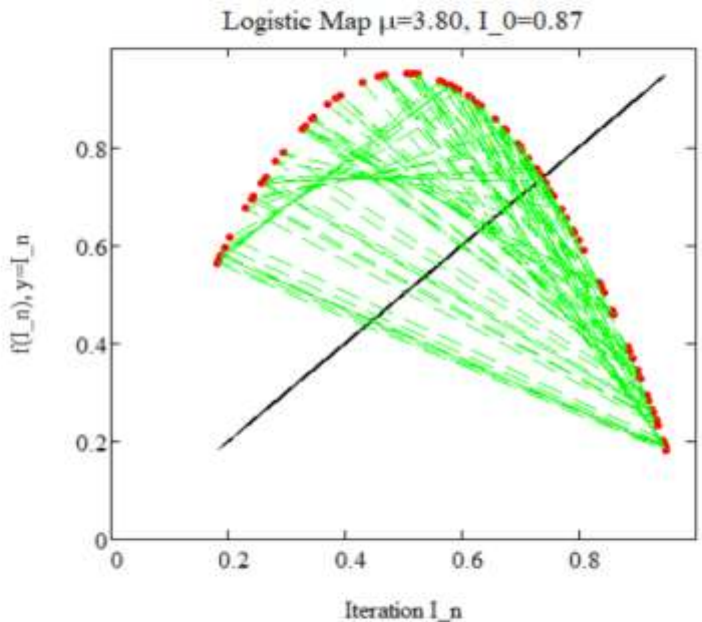
Sensitivity to initial conditions → chaotic operation





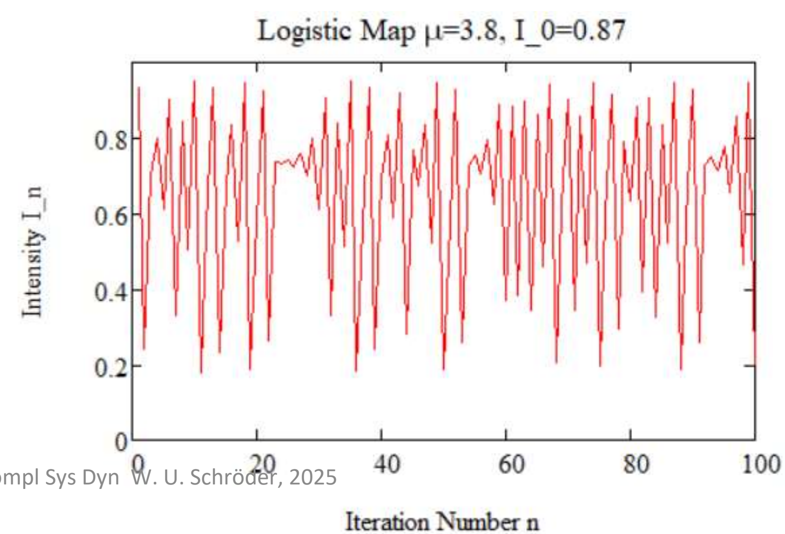
Slightly different I_0 lead to very different time behavior.
N=500 iterations

Chaotic Map Trajectories



Same example as above, plot showing only the iterative intensities I_n on the curve representing the map profile function $f(I)$.

A large part of the brightness spectrum is covered by the trajectory already after 500 iteration. No apparent intensity pattern.
Intensity flashes between bright and dim.



Same example as above, plot shows iterative intensities I_n vs n. Some, but not exact similarities, intermittency domains, strongly dependent on initial condition I_0.

Sensitivity to Initial Conditions

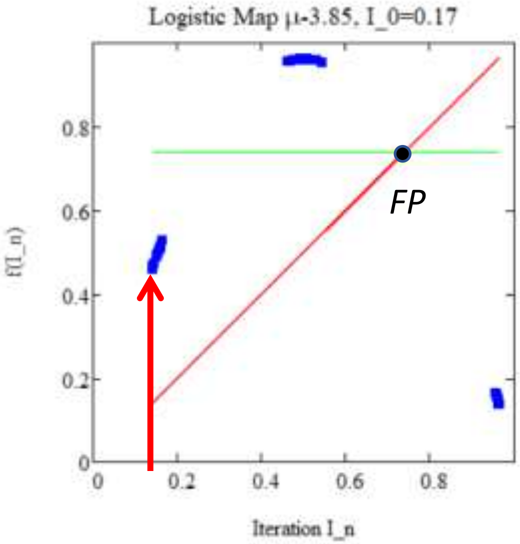


Illustration of sensitivity to initial conditions for

μ = 3.85, fixpoint at *I = 0.74, strange attractor*

IC: I_0 = 0.170, *N* = 100 iterations

Blinking alternatively with 3 different intensities

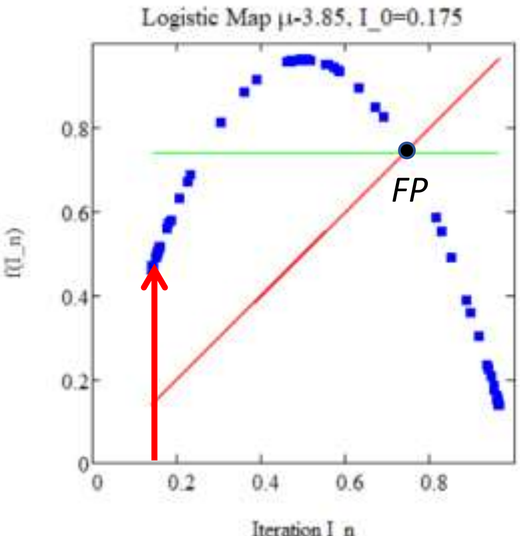


Illustration of sensitivity to initial conditions for

μ = 3.85, fixpoint at *I = 0.74, strange attractor*

IC: I_0 = 0.175, *N* = 100 iterations

Blinking alternatively with a continuum of intensities filling most of the accessible intensity range

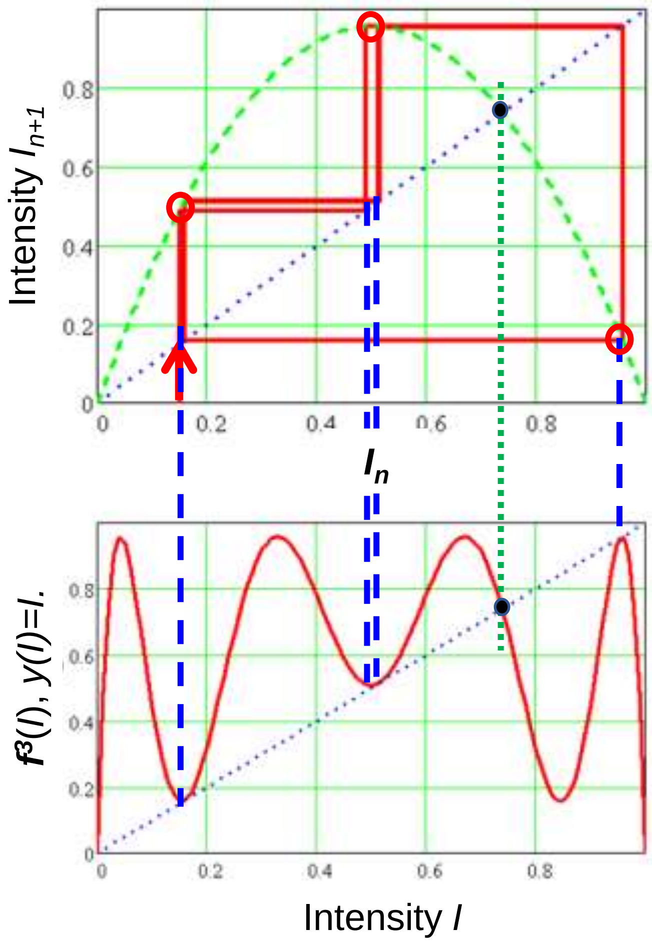
Question: What would the probability distribution look like for the entire chain of N_{total} iterations?

$$I \in \left[I - \frac{\Delta I}{2}, I + \frac{\Delta I}{2} \right) \rightarrow \frac{\Delta P(I)}{\Delta I} := \frac{N(I_n = I)}{N_{total}}; 0 \le I(t) \le 1$$

$$Normalization: P = \sum_{n=1}^{N_{total}} \left(\frac{\Delta P(I_n)}{\Delta I} \right) \cdot \Delta I = 1 \ (= 100\%)$$

Periodic Flashes

Metastable/intermittent processes, strange but predictable trajectories: search for "periodic points." Points of period *n* = stable (attractor) fixpoints of $f^n(x)$.



Fixpoint at $I_f = 0.653$ (black dot) = "strange" attractor:
Trajectory cycles around I_f in *3* periods.

Finding members of strange cycle: look for tangential touching of curve $f^3(\mu, I)$ at $y(I)=I$.

Periodic Flashes

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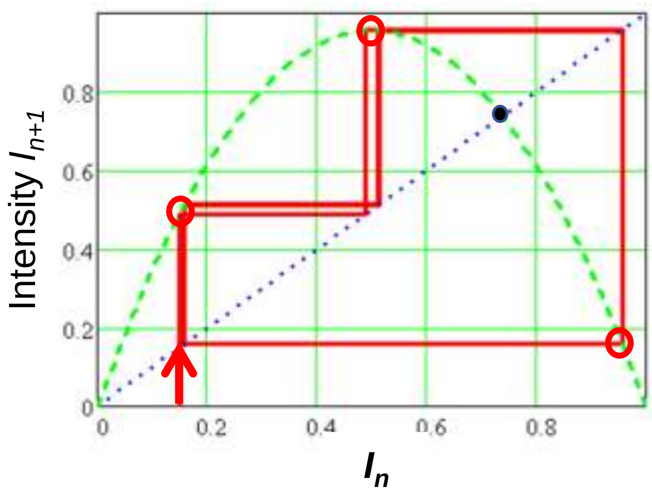

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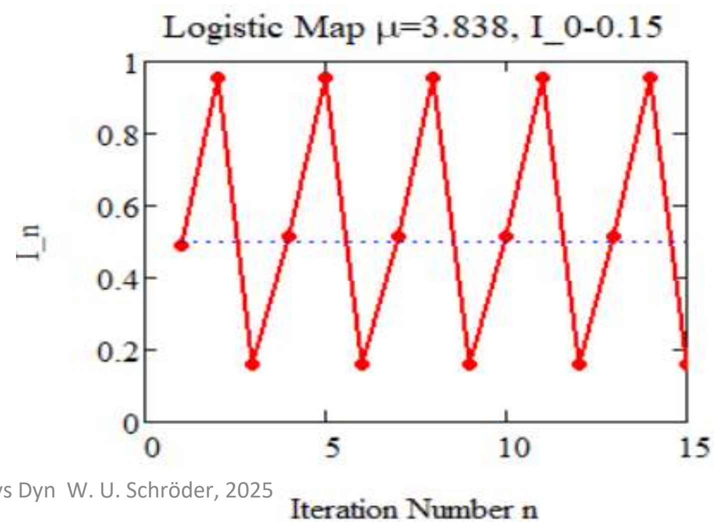


Pattern $f(\mu, I)$ exhibiting periodic triplet blinking patterns : medium, high, low intensity.
Deterministic

Linear and Non-Linear Dynamical Regimes

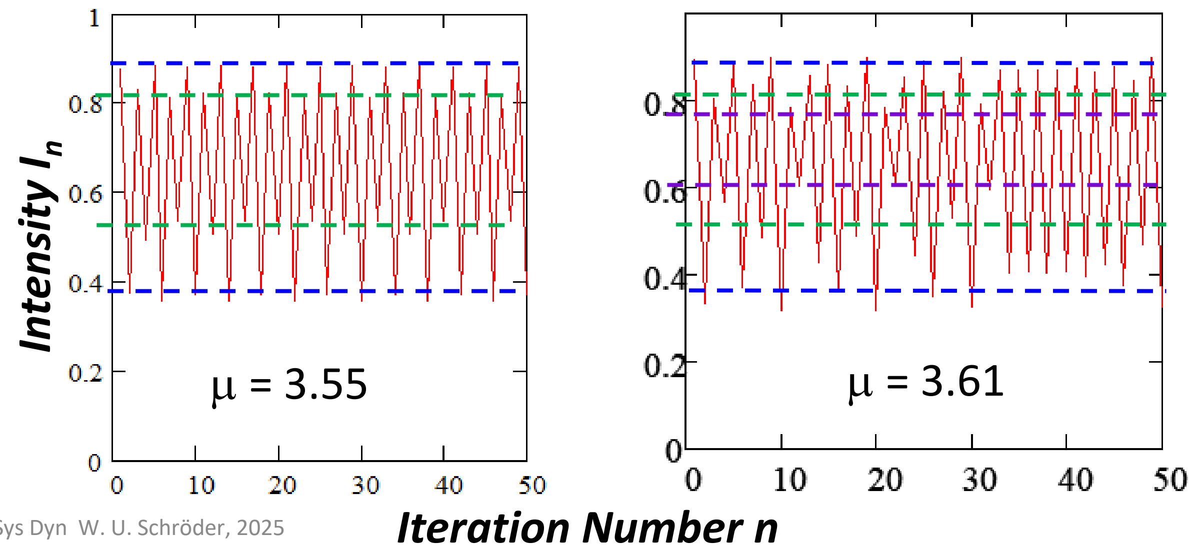
$0.0 \leq \mu \leq 1.0$: *No non – trivial fixpoints* $\rightarrow I_n \underset{n\rightarrow\infty}{\rightarrow} 0$

$1.0 < \mu \leq 3.0$: *1 non – trivial attractor fixpoint, "deterministic chaos"*
Trajectory deterministic for precise initial condition

$3.0 < \mu \leq 3.6$: *1 non – trivial repellor fixpoint, "deterministic chaos"*
bi – stable flickering with alternating intensities,
several n – frequency doublings (bifurcations)

$3.6 < \mu < 3.8$: *1 non – trivial repellor fixpoint, intermittent flicker*

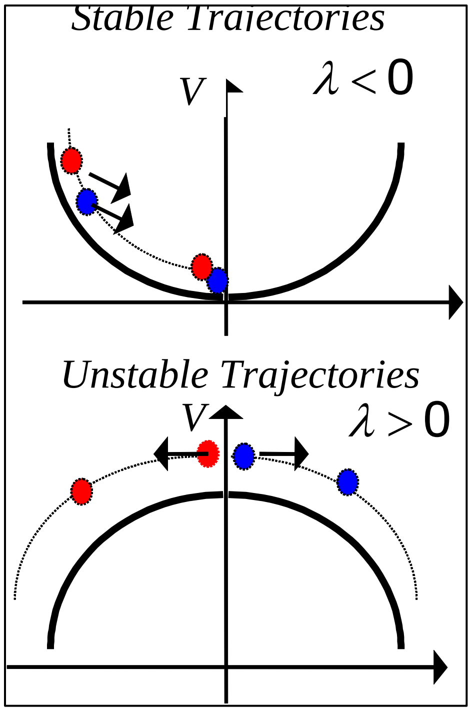
$3.8 \leq \mu < 4.0$: *1 non – trivial repellor fixpoint, chaotic dynamics*



Left: Frequency doubling

Right: Two frequency doublings with intermittency.

Lyapunov Stability Criterion



2 *trajectories initially separated by* $\Delta x = \varepsilon$

Separation at iteration n (time t_n*)*

$$\delta(\varepsilon, n) := | f^n(x) - f^n((x+\varepsilon)) | \quad =: |\varepsilon| \cdot e^{\lambda \cdot n}$$

Stability
$\lambda < 0$

$$Ln\left\{\frac{| f^n(x) - f^n((x+\varepsilon)) |}{\varepsilon}\right\} = \lambda \cdot n \rightarrow \lambda = \frac{1}{n} \cdot Ln\left|\frac{df^n(x)}{dx}\right|$$

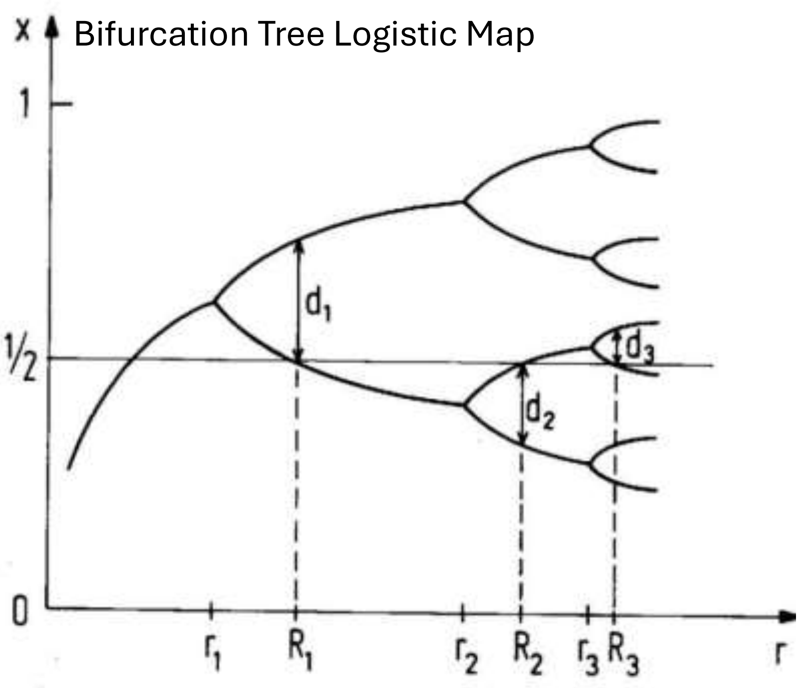
$f^n(x) = f(x_{n-1}) = f(f(x_{n-2}))= \ldots\ldots = f(\ldots\ldots(x_0)\ldots\ldots)$.

$$\frac{df^n}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx_{n-2}} \cdot \frac{dx_{n-2}}{dx} = \ldots\ldots\ldots =$$

$df^n/dx = f'(x_{n-1}) \cdot f'(x_{n-2}) \cdot f'(x_{n-3}) \cdot f'(x_{n-4}) \cdot\cdot f'(x_0)$

$$\lambda = \frac{1}{n} \cdot \sum_{i=0}^{n-1} Ln| f'(x_i) |$$

Tree of Successive Bifurcations



On ordinate $x_{i\to\infty}$ asymptotic iterates (*fix or periodic points* $\boldsymbol{x_n}$, for large *n*) plotted vs. amplification/scaling factor ***r***, for the logistic map. Similar behavior for related maps of the kind

$$f(x) := \mu \cdot x^k \cdot \left(1 - x^k\right)^{1/k};$$

$$\textit{Logistic map } k = 1$$

$$\textit{Feigenbaum Scaling} : \textit{Bifurcation points}$$

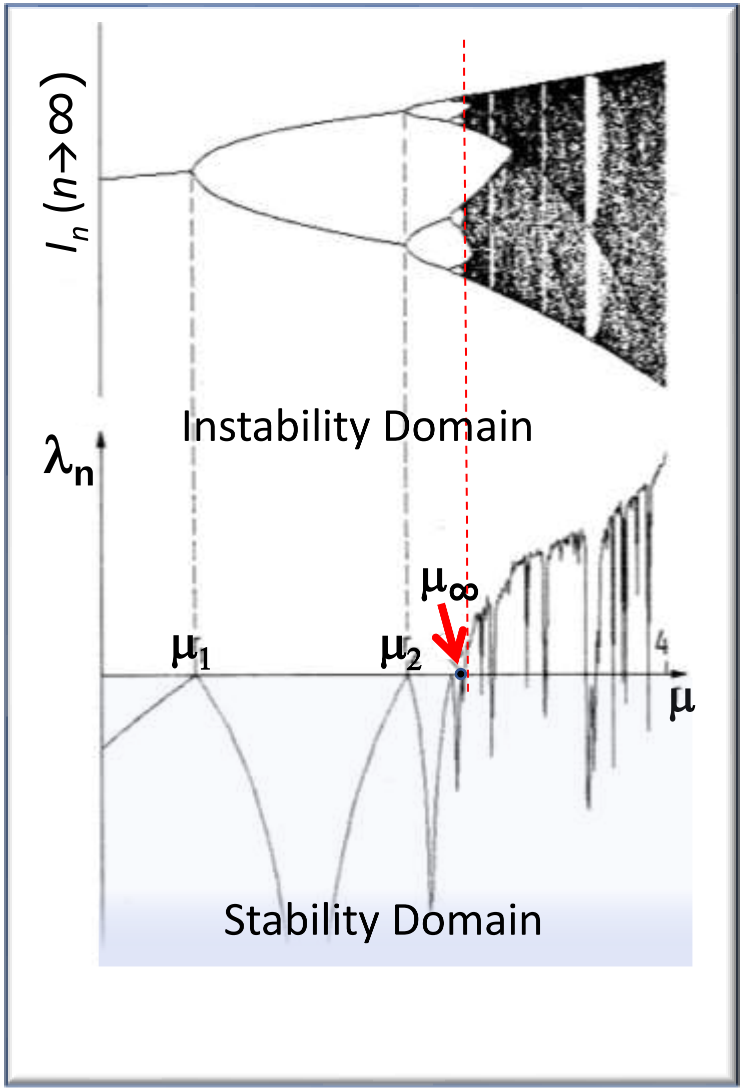
$$r_n = r_\infty - const \cdot \delta^{-n}, n \gg 1$$

$$r_\infty = 3.5699456$$

$$\delta = 4.66920160$$

$$\frac{d_n}{d_{n+1}} = -\alpha, \quad n \gg 1; \alpha = 2.502907875$$

Lyapunov Exponent = $f(\mu)$



Asymptotic iterates and Lyapunov exponent for the logistic map:

Gain factors μ determine dynamics

$\mu \geq \mu_1$: at least 1 bifurcation

$\mu \geq \mu_2$: at least 2 bifurcations

$\mu \geq \mu_\infty$: λ generally >0, → Chaotic system behavior, small special domains for (relatively) orderly behavior.

Similar :

$$f(x) := \mu \cdot x^k \left(1 - x^k\right)^{1/k} \quad \textit{and}$$

$$f(x) := \mu(x) \cdot x^k \left(1 - x^k\right)^{1/k}$$

Outlook and Conclusions (for our environment)

- Non-linear dynamics of complex systems can lead to orderly or chaotic behavior, depending on non-linearity → amplification μ for log. map. strength of positive feed back loops.
- Chaotic dynamics include sudden wild oscillations in system properties at "Tipping Points,"
- Given an observed non-linear behavior for a specific system (example: Earth albedo), it is possible to estimate a Logistic-Map model amplification parameter $\boldsymbol{\mu}$.
- Extensions of simple 1D Logistic-Map model include multiple dimensions {x,y} provide understanding of population dynamics (predator-prey)

$$dx/dt = \mu(x,y) \cdot x \cdot [1-x] \qquad dy/dt = \mu(x,y) \cdot y \cdot [1-y]$$

- Earth albedo can change rapidly, leading to tipping points in climate.

Lyapunov Exponent

$$\boldsymbol{\lambda} = \frac{1}{n} Ln \left| \frac{df^n(x)}{dx} \right|$$

Implicit function

$$f^n(x) = f(x_{n-1}) = \dots = f(f(f(f(x_{n-4}))))\dots$$

$= x_0$

Chain Rule for differentiation:

$$\frac{df^n(x)}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx} = \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{dx_{n-1}}{dx_{n-2}} \cdot \frac{dx_{n-2}}{dx_{n-3}} \cdot \dots \cdot \frac{dx_1}{dx_0}$$

$$= \frac{df(x_{n-1})}{dx_{n-1}} \cdot \frac{df(x_{n-2})}{dx_{n-2}} \cdot \frac{df(x_{n-3})}{dx_{n-3}} \cdot \dots \cdot \frac{df(x)}{dx}$$

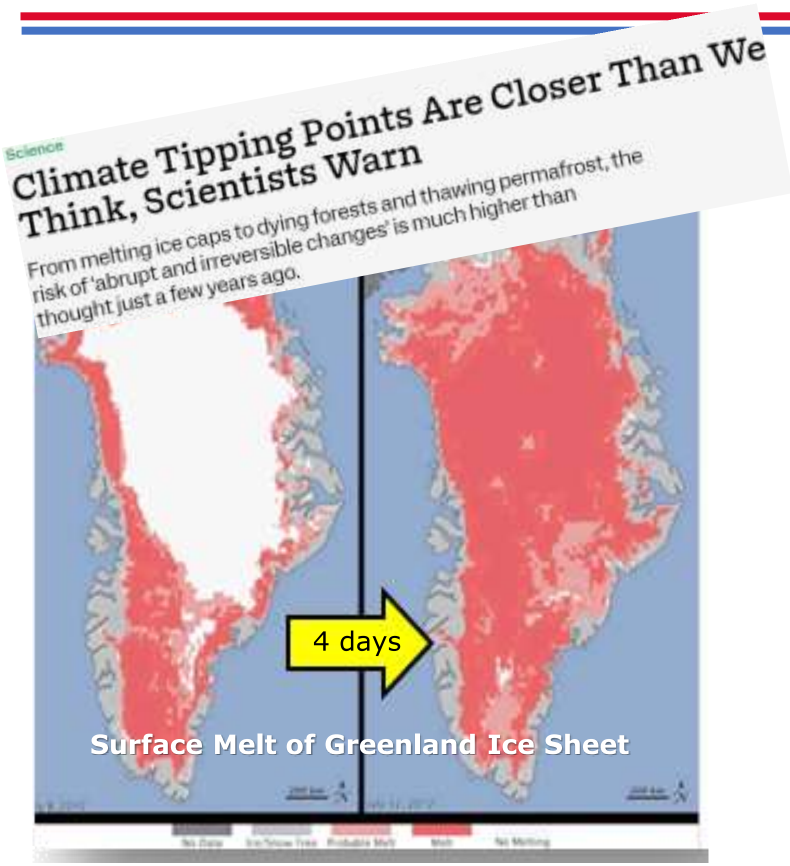
$$Ln \left| \frac{df^n}{dx} \right| = Ln \prod_{i=0}^{n-1} \left| f'(x_i) \right|_{x_i} = \sum_{i=0}^{n-1} Ln \left| f'(x_i) \right|_{x_i}$$

explicit, direct

$$\lambda_n = \frac{1}{n} \cdot \sum_{i=0}^{\mathbf{n}-1} Ln \left| f'(x_i) \right|_{x_i}$$

Large $n \rightarrow \infty$, test the shape of f at many positions x_i

Tipping Points in Earth Climate ?



Non-linear and coupled effects in Earth current climate evolution → global warming, melting of sea ice , ice cap, desertification, ocean acidification, sea level rise,……

Historic climate facts:

Earth climate has alternated between **Ice ages** (little and major) and **greenhouse** periods. Transition speed?

Do we have time to adapt or change pace?

Mind the fate of planet Venus (NYT 012921)

Earth albedo or surface reflectivity $\boldsymbol{\varepsilon}$ = important in maintaining radiation balance

Glaciation: increasing ice cover $\Delta\varepsilon > 0 \rightarrow$ *surface temperature change* $\Delta T < 0$

Warming: decreasing ice cover $\Delta\varepsilon < 0 \rightarrow$ *surface temperature change* $\Delta T > 0$

Albedo is non-monotonic function of important driving parameters, has extrema!

Earth Albedo Model

Albedo is non-monotonic function of important driving parameters.

Combine ε parameter dependence to model *non-linear* dependence on history:

$$\varepsilon\left(t+\Delta t\right)=\alpha\cdot\varepsilon\left(t\right)-\beta\cdot\varepsilon^{2}\left(t\right)+......;\quad parameters\ \alpha,\beta=f(CO_{2},...)?$$

$$Since\ \varepsilon\left(t\right)\ is\ non-monotonic\ and\ must\ have\ an\ extremum$$

$$\rightarrow sign\left(\alpha\right)=sign\left(\beta\right),\ choose\ \alpha,\beta>0$$

$$Adopt\ discrete\ time\ steps\ t_{n}\ \ (days,\ months,\ years,....,centuries)\rightarrow$$

$$\varepsilon_{n+1}=\varepsilon_{n}\left(t+n\cdot\Delta t\right)\approx\alpha\cdot\varepsilon_{n}-\beta\cdot\varepsilon_{n}^{2}\qquad \text{"Iteration"}$$

$$Variable\ transformation\rightarrow$$

$$Profile\ function\ \boxed{f(\varepsilon)=\mu\cdot\varepsilon\cdot\left(1-\varepsilon\right)}\qquad \text{"Logistic Map"}$$

$$\varepsilon_{n+1}=f(\varepsilon_{n})=f(f(\varepsilon_{n-1}))=f(f(f(\varepsilon_{n-2})))=f^{3}(\varepsilon_{n-2})\quad Iterative\ Logistic\ Map$$

Intermediate Summary

Linear force laws are deterministic → lead to predictable time evolution and are not sensitive to initial conditions: Small changes in initial conditions → small changes in final positions and momenta

$$f\left(x+\Delta x\right)\approx f\left(x\right)+\Delta x\cdot f'\left(x\right)$$

- Complex, unpredictable (Chaotic) dynamics can be caused by non-linear force laws *for certain complex systems*, which have specific sets of properties (→model parameters).
- Complex, unpredictable (Chaotic) dynamics can be caused by correlated motion along different degrees of freedom. Rate equations for different d.o.f. become substantially entangled.
- Predictable ("orderly") dynamics is characterized by insensitivity to initial conditions.
- Unpredictable ("chaotic") dynamics is associated with high sensitivity to initial conditions.
- Both, orderly and chaotic dynamics can lead to asymptotically ($t \rightarrow \infty$) predictable states (deterministic chaos).
- Chaotic dynamics can lead to different classes of periodic or (quasi-) random asymptotic states.

Important examples: global climate, biological population dynamics (intra/inter-species competition), organ functionality, catalytic chemical reactions.

Analyze a simple (1D) chaotic system (climate rad balance, electric circuits)